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Modeling flocks and prices: Jumping particles with an attractive interaction

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Abstract. We introduce and investigate a new model of a finite number of particles jumping forward on the real line. The jump lengths are independent of everything, but the jump rate of each particle depends on the relative position of the particle compared to the center of mass of the system. The rates are higher for those left behind, and lower for those ahead of the center of mass, providing an attractive interaction keeping the particles together. We prove that in the fluid limit, as the number of particles goes to infinity, the evolution of the system is described by a mean field equation that exhibits traveling wave solutions. A connection to extreme value statistics is also provided.

Résumé. Nous introduisons et étudions un nouveau modèle comprenant un nombre fini de particules situées sur la droite réelle et pouvant effectuer des sauts vers la droite. Les longueurs des sauts sont indépendantes du reste, mais le taux de saut de chaque particule dépend de la position relative de la particule par rapport au centre de masse du système. Les taux sont plus grands pour celles qui sont en retard, et plus petits pour celles qui sont en avance par rapport au centre de masse; cela crée ainsi une interaction attractive qui favorise la cohésion des particules. Nous montrons qu'à la limite fluide, lorsque le nombre de particules tend vers l'infini, l'évolution du système est décrite par une équation de champ moyen ayant des solutions d'ondes progressives. On présente également un lien avec les statistiques des valeurs extrêmes.

1. Introduction

Interacting particle systems always show interesting behavior and present a challenge for rigorous treatment. In this paper we introduce and investigate a new type of model which – to the best of our knowledge – has not emerged in the literature before. The system consists of a finite number of particles moving forward on the real line $\mathbb{R}$. The center of mass of these particles is computed and, given a configuration, each particle jumps after an exponential waiting time that depends on its relative position compared to the center of mass. The more a particle lags behind the center of mass, the less its mean waiting time is; on the other hand, the more a particle is ahead of the center

mass, the more its mean waiting time is. When the exponential time has passed, the particle jumps forward a random distance, independent of everything else, drawn from a probability distribution concentrated on the non-negative half-line. While doing so, it naturally moves the center of mass forward by $1/n$ times its jump length as well, where n is the number of particles. Thus the dynamics has an effect of keeping the particles together, while randomness creates fluctuations in the system.

Several real-life phenomena can be modeled this way: the evolution of prices set by agents trying to sell the same product in a market, the position of goats in a herd, etc. Mathematical finance is an area of active research, and many processes arising in finance are modeled with interacting particle systems, see e.g. the monograph on stochastic portfolio theory by Fernholz [15], Banner, Fernholz and Karatzas [5], and the survey by Fernholz and Karatzas [16]. Herding behavior has drawn serious interest in the physics literature, a few examples are Vicsek, Czirók, Ben–Jacob, Cohen and Shochet [32], Czirók, Barabási and Vicsek [11], Ballerini, Cabibbo, Candelier, Cavagna, Cisbani, Giardina, Lecomte, Orlandi, Parisi, Procaccini, Viale and Zdravkovic [4].

The problem of competing individuals is interesting even without dependence of the dynamics on the actual position of the particles. A series of papers have been written on models where the competing particles evolve according to i.i.d. increments: Ruzmaikina and Aizenman [27], Arguin [1], Arguin and Aizenman [2], Shkolnikov [28]. The main point of investigation in these papers is characterizing point processes for which the joint distribution of the gaps between the particles is invariant under the evolution.

Several authors have considered continuous models in which the motion (drift) of the particles depends on their rank in the spatial ordering; see e.g. Banner, Fernholz and Karatzas [5], Pal and Pitman [25], Chatterjee and Pal [9], Shkolnikov [29–31]. These models are central objects of study in modeling capital distributions in equity markets. Branching diffusion-type models have also been investigated with a linear interaction between particles, see e.g. Greven and Hollander [19], Engländer [12] and the references therein. Linearity of the interaction is an important ingredient in these papers making the analysis somewhat easier.

Closer to the present work are jump processes with interaction between the jumping particles. A related model is introduced in ben-Avraham, Majumdar and Redner [6], where particles compete with a leader who looks back only at the contestant directly behind him. Interaction between the leader and the followers is realized via a linear dependence of the mean jump length on the relative position of the leader and the followers. A process with rank-dependent jump rates is investigated in Greenberg, Malyshev and Popov [18], and one with similar, but randomized jump rates in Manita and Shcherbakov [24]. Maybe the closest model to our system is the one featured in Grigorescu and Kang [20], where particles jump in a configuration-dependent manner, and the authors give special focus to the situation where the dependence of particle advances on the configuration is via the center of mass of the particles.

In our case, positive jump lengths are independently chosen, and the jump rate of a particle depends on its position relative to the center of mass. This *jump rate function* is non-increasing and non-negative, hence clearly cannot be linear in nontrivial cases. The setting is thus genuinely different from situations in the above examples. The non-increasing jump rate function creates a monotonicity in the system which keeps the particles together. This gives rise to a traveling wave-type stationary distribution. We address the following questions.

- The two-particle case is, of course, tractable. We compute the stationary distribution of the gap between the two particles for several choices of the jump length distribution. However, we cannot handle the situation even for three particles. (The process is not reversible.)
- As the number of particles goes to infinity, we expect that the motion of the center of mass converges to a deterministic one. This motion, together with the distribution of the particles, satisfy a deterministic integro-differential equation in this so-called *fluid limit*. We call the resulting equation the mean field equation. We write up this equation by heuristic arguments and, in some cases, we solve it for a traveling wave form, i.e. the stationary solution as seen from the center of mass. Even finding the traveling wave solution to this mean field equation is nontrivial.

    The reason for expecting a deterministic limiting equation is the averaging property of the dynamics. We in fact prove this phenomena in this paper later on for some cases. As the motion of the center of mass becomes deterministic, the particles decouple, each independently following the deterministically moving center of mass. Therefore in the limit it does not matter if we consider a deterministic cloud of particles (the limit coming from the empirical distribution of $n \to \infty$ particles), or a single particle in the corresponding distribution that is driven by the deterministic center of mass. The evolution of the fluid limit can therefore be called a mean field dynamics. Of course, the motion of the center of mass must be chosen appropriately to be consistent with the cloud or distribution

of particles. In the traveling wave case, the distribution, as seen from the center of mass, becomes stationary, and the speed of the center of mass becomes constant in time.

- Some traveling wave solutions lead us to Gumbel distributions, which naturally pose the question whether an extreme value theory interpretation can be given to the process. An idea by Attila Rákos shows that there is indeed such a connection.
- Under restrictive assumptions on the jump rates, we prove that the evolution of the process in the fluid limit indeed satisfies the deterministic integro-differential equation we formulated by heuristic arguments. We use weak convergence methods and uniqueness results on the integro-differential equation. Since the center of mass is not a bounded function of the particle positions, standard weak convergence techniques need to be extended to our situation.

This last point is the most interesting from a mathematical point of view. Our proof is a standard fluid limit argument, somewhat similar to those in Manita and Shcherbakov [24] and Grigorescu and Kang [20]: we use an extended notion of weak convergence, tightness, and martingales to show convergence to the unique solution of the limiting integro-differential equation. As opposed to Greenberg, Malyshev and Popov [18], asymptotic independence of the particles (as the number of particles tends to infinity) will be a consequence of the proof, rather than a method to rely on. This is because our case of dependence through the center of mass does not seem to provide an easy a priori way of showing asymptotic independence of the particles.

We start by precisely describing the model in Section 2. We formulate our main results in Section 3: first, we formulate the limiting mean field equation together with some traveling wave solutions, and we also state the fluid limit theorem. These are the results we believe to be the most important, and additional statements can be found in subsequent sections. We end this section with a collection of open questions. We proceed by analyzing the two-particle case in Section 4. The mean field equation is investigated in Section 5. In particular, a connection with extreme value statistics is heuristically explained in Section 5.2. Finally, Section 6 contains our fluid limit argument under some restrictive conditions on the jump rates.

The reader may find an extended version of this paper in [3], where we provide more details on arguments that are not essential for our main results, particularly in Sections 4 and 5. In addition, see [3], Section 7 for simulation results which support the idea that the mean field equation holds in the fluid limit in greater generality.

2. The model

To model the phenomena described in the Introduction, we present the following model. The model consists of n particles on the real line. Let us denote the position of the particles by $x_1(t), x_2(t), \ldots, x_n(t)$, where t represents time and $x_i(t) \in \mathbb{R}$ for $i = 1, \ldots, n$. We now describe the dynamics. Given a configuration, the mean position (center of mass) of the particles is $m_n(t) = \frac{1}{n} \sum_{i=1}^{n} x_i(t)$. Now let us denote by $w : \mathbb{R} \to \mathbb{R}^+$ the *jump rate function*, a positive and monotone decreasing function. The dynamics is a continuous-time Markov jump process. The ith particle, positioned at $x_i(t)$ at time t, jumps with rate $w(x_i(t) - m_n(t))$. That is, conditioned on the configuration of the particles, jumps happen independently after an exponentially distributed time, with the parameter of the ith particle being $w(x_i(t) - m_n(t))$.

When a particle jumps, the length of the jump is a random positive number Z from a specific distribution. This is independent of time and the position of the particle and all other particles as well. We scale the jump length so that $\mathbb{E}Z = 1$, and for technical reasons we assume that Z has a finite third moment. If this distribution is absolutely continuous then its density is denoted by φ.

Our focus is on the *empirical measure* of the n particle process at time t:

$$\mu_n(t) = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i(t)}.$$

More generally, we are interested in the *path* of the empirical measure, $\mu_n(\cdot) = (\mu_n(t))_{t \geq 0}$, with initial condition $\mu_n(0) = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i(0)}$.

We introduce the following notation in order to abbreviate formulas, which is also well suited for weak convergence type arguments. For a function $f : \mathbb{R} \to \mathbb{R}$ and a probability measure P on $\mathbb{R}$ let

$$\langle f, P \rangle := \int_{-\infty}^{\infty} f(x) P(\mathrm{d}x).$$

We also use the following notation: $\langle f(x), P \rangle \equiv \langle f, P \rangle$. Using this notation, we have $m_n(t) = \langle x, \mu_n(t) \rangle$.

Now we can present the generator of the process acting on the integrated test functions as above. Let C_b denote the space of bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and let Id denote the identity from $\mathbb{R}$ to $\mathbb{R}$: $\mathrm{Id}(x) = x$. Define

$$H := \{ f \in C_b \colon |f| \le 1 \} \cup \{ \mathrm{Id} \}. \tag{2.1}$$

This set of functions will be important later on. For a function $f \in H$ we have:

$$L\langle f, \mu_n(t) \rangle = \langle (\mathbb{E}(f(x+Z)) - f(x)) w(x - m_n(t)), \mu_n(t) \rangle,$$

where L is the generator of the process. The particular choice of H is so that we can apply the generator L to the center of mass $m_n(t)$.

3. Results

Our main contributions and results are (1) introducing the model, (2) formulating via heuristics the limiting behavior as the number of particles goes to infinity and analyzing this limiting behavior, and (3) showing rigorously that (under some assumptions on the jump rates) the limiting behavior of the process in the fluid limit indeed satisfies the deterministic integro-differential equation that we formulated. This section describes in more detail results describing (2) and (3).

The case of a fixed number n of particles is difficult to treat. Of course the $n = 2$ case is doable: in Section 4 we look at two specific cases of the model, and in both we compute the stationary distribution of the gap between the two particles. At the end of the section we illustrate the difficulties that arise for more than two particles in Fig. 2.

Our main interest is in the large n limit (so called *fluid limit*) of the particle system. Let us assume that the jump length distribution has a density φ. Heuristically, in the fluid limit the empirical measure $\mu_n(t)$ of the particles at time t becomes an absolutely continuous measure with density $\varrho(\cdot, t)$ whose time evolution is given by the so-called mean field equation

$$\frac{\partial \varrho(x,t)}{\partial t} = -w(x - m(t)) \varrho(x,t) + \int_{-\infty}^{x} w(y - m(t)) \varrho(y,t) \varphi(x-y) \, \mathrm{d}y, \tag{3.1}$$

where

$$m(t) = \int_{-\infty}^{\infty} x \varrho(x,t) \, \mathrm{d}x \tag{3.2}$$

is the mean of the probability density $\varrho(\cdot, t)$. A priori we assume nothing of the density $\varrho(\cdot, t)$ other than it having a finite mean. The first term on the right-hand side of (3.1) accounts for the particles hopping forward from position x at time t: at time t there is a density of $\varrho(x,t)$ particles at x, who jump forward with rate $w(x - m(t))$. The second, integral term on the right-hand side of (3.1) accounts for the particles who are at position $y < x$ jumping forward at time t to position x: at time t there is a density of $\varrho(y,t)$ particles at y, who jump forward with rate $w(y - m(t))$, and need to jump forward exactly $x - y$ to land at x, the probability of this being proportional to $\varphi(x - y)$.

Our first results concern solutions to the mean field equation (3.1) in the form of a traveling wave

$$\varrho(x,t) = \rho(x - ct), \tag{3.3}$$

where c is the constant velocity of the wave and ρ is centered, i.e. $\int x \rho(x) \, \mathrm{d}x = 0$. Traveling waves of this form represent stationary distributions as seen from the center of mass, which travels at a constant speed c. Substituting (3.3) into the mean field equation (3.1), and then writing x instead of $x - ct$ for simplicity everywhere, we arrive at the following equation:

$$-c\rho'(x) = -w(x)\rho(x) + \int_{-\infty}^{x} w(y)\rho(y)\varphi(x-y) \, \mathrm{d}y. \tag{3.4}$$

Recall that the jump rate function w is a non-increasing function; any non-constant w gives a strong enough attraction to keep the particles together, i.e. there is a probability density traveling wave solution ρ. This is formulated in the following theorem for exponentially distributed jump lengths.

Theorem 3.1. *Consider the specific case when the jump length distribution is exponential with mean one, and assume that w is non-constant. Let c_0 be the unique solution of*

$$\int_{-\infty}^{\infty} x e^{\int_0^x ((1/c_0)w(s)-1)\,\mathrm{d}s}\,\mathrm{d}x = 0. \tag{3.5}$$

Equation (3.4) *has a centered probability density solution ρ if and only if $c = c_0$. In this case it is given by*

$$\rho(x) = K e^{\int_0^x ((1/c)w(s)-1)\,\mathrm{d}s}, \tag{3.6}$$

where K is an appropriate normalizing constant. It is immediate that the probability density ρ of (3.6) *has at least exponentially decreasing tails.*

There are several specific jump rate functions w for which the resulting stationary distribution is of interest. In particular, the case of $w(x) = e^{-\beta x}$, where $\beta > 0$, leads to the generalized Gumbel distribution:

Corollary 3.2. *If $w(x) = e^{-\beta x}$, where $\beta > 0$, then from Theorem* 3.1 *we obtain for the stationary density*

$$\rho_\beta(x) = \frac{\beta}{\Gamma((1/\beta))} e^{-(x-\psi(1/\beta)/\beta)-e^{-\beta(x-\psi(1/\beta)/\beta)}}, \tag{3.7}$$

where ψ is the digamma function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Specifically for $\beta = 1$ the stationary distribution is the centered standard Gumbel distribution. The velocity of the wave is

$$c = \frac{1}{\beta} e^{-\psi(1/\beta)}.$$

The Gumbel distribution often arises in extreme value theory (Gumbel [21]), e.g. this is the limiting distribution of the rescaled maximum of a large number of independent identically distributed random variables drawn from a distribution that has no upper bound and decays faster than any power law at infinity (e.g. exponential or Gaussian distributions). More generally, the kth largest value, once rescaled, follows the generalized Gumbel distribution (Bertin [7], Clusel and Bertin [10], Gumbel [21]). Therefore the question arises whether there is an underlying extremal process in the dynamics. It turns out that there is a connection between the mean field model and extreme value theory: this is discussed in Section 5.2.

The validity of the mean field equation is strongly supported by simulation results for various special cases of the model, see [3], Section 7. We expect this to be true in quite some generality (for fairly general jump rate functions and jump length distributions).

The main result of the paper is to rigorously validate this approximation. That is, our goal is to show that in the fluid limit – when the number of particles tends to infinity – the mean field equation (3.1) holds. However, when trying to rigorously prove this, technical difficulties may arise. Therefore we impose some restrictions on the jump rate function and the jump length distribution. Our main restriction is that we suppose that w is bounded: $w(x) \le a$. We also impose some additional (smoothness) restrictions on w, but boundedness is the main restriction. Our restrictions on the jump length distribution are light: we assume that it has a finite third moment.

In order to precisely state our main theorem, some preparations are needed. First notice that the mean field equation (3.1) can be rewritten as the evolution of a probability measure $\mu(\cdot)$ using test functions from H. Moreover, we do not have to assume that the jump length distribution has a density either. Define

$$A_{t,f}\big(\mu(\cdot)\big) := \langle f, \mu(t)\rangle - \langle f, \mu(0)\rangle - \int_0^t \big\langle \big(\mathbb{E}\big(f(x+Z)\big) - f(x)\big) w\big(x - m(s)\big), \mu(s)\big\rangle \mathrm{d}s$$

$$= \langle f, \mu(t)\rangle - \langle f, \mu(0)\rangle - \int_0^t L\langle f, \mu(s)\rangle \mathrm{d}s, \tag{3.8}$$

where

$$m(t) = \langle x, \mu(t)\rangle.$$

We say that $\mu(\cdot)$ satisfies the mean field equation if

$$A_{t,f}\big(\mu(\cdot)\big) = 0 \tag{3.9}$$

for every $t \geq 0$ and $f \in H$.

The position of the center of mass of the particles plays an important role in our model, and therefore we always assume it exists (and is finite). Consequently, it is natural to look at $\mu_n(t)$ as an element of $\mathcal{P}_1(\mathbb{R})$, the space of probability measures on $\mathbb{R}$ having a finite first (absolute) moment. In order to talk about convergence in $\mathcal{P}_1(\mathbb{R})$ we need to define a metric on $\mathcal{P}_1(\mathbb{R})$. We use the so-called *1-Wasserstein metric*. For $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ define the set of coupling measures

$$\Pi(\mu, \nu) := \big\{\pi \in M_1(\mathbb{R} \times \mathbb{R}): \pi(\cdot \times \mathbb{R}) = \mu, \pi(\mathbb{R} \times \cdot) = \nu\big\},$$

where $M_1(\mathbb{R} \times \mathbb{R})$ denotes the space of probability measures on $\mathbb{R} \times \mathbb{R}$. The 1-*Wasserstein metric* on the space $\mathcal{P}_1(\mathbb{R})$ is defined as

$$d_1(\mu, \nu) := \inf\bigg\{\int_{\mathbb{R} \times \mathbb{R}} |x - y| \pi(\mathrm{d}x, \mathrm{d}y): \pi \in \Pi(\mu, \nu)\bigg\} \tag{3.10}$$

for $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$. Much is known about the metric space $(\mathcal{P}_1(\mathbb{R}), d_1)$; we will state what we need later.

The natural space in which to consider $\mu_n(\cdot)$ is the Skorohod space $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$: we know that $\mu_n(\cdot)$ is a piece-wise constant function in $\mathcal{P}_1(\mathbb{R})$, having jumps whenever a particle jumps. Similarly, for any test function f (for instance $f \in H$), the natural space in which to consider $\langle f, \mu_n(\cdot)\rangle$ is the Skorohod space $D([0, \infty), \mathbb{R})$.

In order to prove weak convergence of $\{\mu_n(\cdot)\}_{n \geq 1}$ in the appropriate Skorohod space, we have to make the following assumptions on the initial distribution of the particles.

Assumption 1. *Our first assumption is that*

$$\lim_{L \to \infty} \limsup_{n \to \infty} \mathbb{P}\bigg(\frac{1}{n}\sum_{i=1}^n |x_i(0)| > L\bigg) = 0. \tag{3.11}$$

Assumption 2. *Our second assumption is that there exists a constant $\hat{c}$ such that for every $\varepsilon > 0$*

$$\sup_n \mathbb{P}\bigg(\frac{1}{n}\sum_{i=1}^n \mathbf{1}\big[|x_i(0)| > \hat{c}\varepsilon^{-2/3}\big] \geq \varepsilon/4\bigg) \leq \varepsilon/3.$$

(Note: the exponent 2/3 has no special significance. This is simply because we assume $\mathbb{E}(Z^3) < \infty$ for simplicity. We could assume $\mathbb{E}(Z^{2+\epsilon}) < \infty$ for some small $\epsilon > 0$, in which case we could replace the exponent 2/3 with an exponent of $2/(2 + \epsilon)$ in Assumption 2.)

A sufficient and simple condition for both Assumption 1 and 2 to hold is that

$$\sup_n \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}|x_i(0)|^3\right) < \infty.$$

Assumption 3. *Our third assumption is that*

$$\mu_n(0) \Rightarrow_n \nu \tag{3.12}$$

in $\mathcal{P}_1(\mathbb{R})$, *where* $\nu \in \mathcal{P}_1(\mathbb{R})$ *is a deterministic initial profile and* $\Rightarrow_n$ *denotes weak convergence.*

We are now ready to state our main theorem.

Theorem 3.3. *Suppose that the non-increasing jump rate function* w *is bounded, and that it is differentiable with bounded derivative. Suppose the jump length* Z *is scaled such that* $\mathbb{E}Z = 1$ *and that it has a finite third moment. Suppose that Assumptions* 1, 2 *and* 3 *from above hold. Then*

$$\mu_n(\cdot) \Rightarrow_n \mu(\cdot) \tag{3.13}$$

in $D([0,\infty), \mathcal{P}_1(\mathbb{R}))$, *where* $\mu(\cdot)$ *is the unique deterministic solution to the mean field equation* (3.9) *with initial condition* ν.

The proof of this theorem is in Section 6. The argument follows the standard method for proving weak convergence results. Namely, based on Prohorov's theorem, we need to do three things:

- Prove tightness of $\{\mu_n(\cdot)\}_{n\geq 1}$ in the Skorohod space $D([0,\infty), \mathcal{P}_1(\mathbb{R}))$.
- Identify the limit of $\{\mu_n(\cdot)\}_{n\geq 1}$: here we essentially have to show that the limit solves the deterministic mean field equation.
- Prove uniqueness of the solution to the mean field equation with a given initial condition.

3.1. *Open questions*

We end this section with some open questions.

- Theorem 3.3 is a law of large numbers type result, so it is natural to ask about the fluctuations around this first order limit. In particular, how does the variance of the empirical mean $m_n(t)$ scale with n and t? Preliminary simulations suggest that this variance grows linearly in t, and for fixed t decays in n as $n^{-\alpha}$ for some $\alpha > 0$. (The simulations are inconclusive of what the exponent α should be.)
- Once the order of fluctuations of the empirical mean $m_n(t)$ is known, can we determine the limiting distribution of $m_n(t)$ with appropriate scaling?
- Going back to the finite particle system: can we really not deal with a fixed number $(n \geq 3)$ of particles and determine the stationary distribution viewed from the center of mass in this case?
- In Theorem 3.3 we make assumptions on the jump rate function w, the jump length distribution, and the initial distribution of the particles. Most of the assumptions are fairly weak, except for the assumption that w is bounded, which is the main restrictive assumption. We believe that these assumptions can be weakened, and in particular we are interested in removing the assumption that w is bounded. Currently all three steps of the proof (tightness, identifying the limit, uniqueness of the limit) use estimates which involve the bound on w. One crucial thing that would be needed is control over the probability of a particle "falling behind" by a lot. Can one prove the same result for unbounded w, e.g. $w(x) = e^{-x}$?

4. Two-particle system

As an introduction to the model, we look at the simplest case: when $n = 2$, i.e. when there are two competing particles. In particular, we look at two specific cases of the model, and in each case we compute the stationary distribution of the gap between the two particles. At the end of the section we illustrate the difficulties that arise in the case of $n \geq 3$ particles.

4.1. *Deterministic jumps*

First, let us look at the model when the length of the jumps is not random but a deterministic positive number. We always have the freedom to scale the system so that this length is equal to unity. So, in this subsection, we look at the model with deterministic jumps of length 1.

In the case of two particles, $x_1(t)$ and $x_2(t)$, we are interested in how far away they move from each other, thus in the gap $g(t) = |x_1(t) - x_2(t)|$. For simplicity let us suppose that $x_1(0) - x_2(0) \in \mathbb{Z}$, so that $g(t) \in \mathbb{N}$ for all $t \geq 0$. In this case $g(t)$ is a continuous-time Markov process on $\mathbb{N}$, with infinitesimal generator Q given by the model in the following way. Since the gap can only change by 1, Q is a tridiagonal matrix. If the gap is 0, i.e. $x_1(t) = x_2(t)$, then both particles jump with rate $w(0)$ and whichever one does, the gap changes to 1. Therefore the rate at which the gap changes from 0 to 1 is $2w(0)$ and thus

$$Q_{0,0} = -2w(0), \qquad Q_{0,1} = 2w(0).$$

If the gap between the particles is $k > 0$, then the one in the lead jumps with rate $w(\frac{k}{2})$, while the one which is behind jumps with rate $w(-\frac{k}{2})$. This means that the gap increases by 1 with rate $w(\frac{k}{2})$ and decreases by 1 with rate $w(-\frac{k}{2})$ and so

$$Q_{k,k-1} = w\left(-\frac{k}{2}\right), \qquad Q_{k,k} = -\left(w\left(-\frac{k}{2}\right) + w\left(\frac{k}{2}\right)\right), \qquad Q_{k,k+1} = w\left(\frac{k}{2}\right).$$

When the jump rate function w is a monotone decreasing, non-constant function, then $g(t)$ is a positive recurrent birth-and-death process with stationary distribution $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$ given by

$$\pi_0 = \frac{1}{1 + 2\sum_{k=1}^{\infty} \prod_{i=0}^{k-1} w(i/2)/\prod_{i=1}^{k} w(-i/2)},$$

$$\pi_k = 2\pi_0 \frac{\prod_{i=0}^{k-1} w(i/2)}{\prod_{i=1}^{k} w(-i/2)} \quad \text{for } k \geq 1.$$

In particular, if the jump rate function is exponential, i.e. $w(x) = e^{-\beta x}$, where $\beta > 0$, then the stationary distribution shows Gaussian decay, while if the jump rate function is a step function,

$$w(x) = \begin{cases} a & \text{if } x < 0, \\ b & \text{if } 0 \leq x, \end{cases}$$

where $a > b > 0$, then the stationary distribution shows exponential decay; see [3], Section 4.1.

4.2. *Jump length distribution having a density*

Consider again the case of two particles, but now let us suppose that the distribution of the length of the jumps has a density, and denote this by φ as mentioned in Section 2. As before, we are interested in the gap $g(t) = |x_1(t) - x_2(t)|$ between the two particles. Denote by $p(g, t)$ the probability density of the gap g at time t, which is concentrated on $[0, \infty)$. The time evolution of the gap is completely described by the time evolution of p, which is described by the
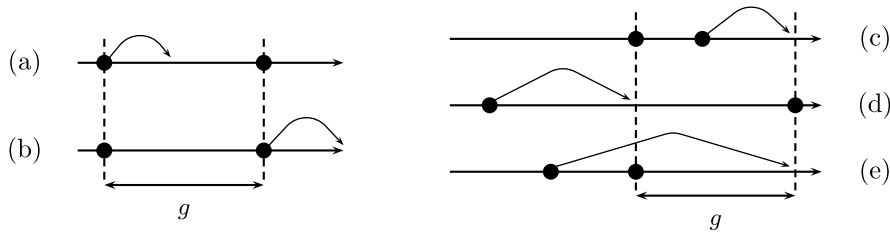
Fig. 1. Events that change the gap length.

master equation

$$\dot{p}(g) = -p(g)\big(w(-g/2) + w(g/2)\big) + \int_0^g p(y)w(y/2)\varphi(g-y)\,\mathrm{d}y$$

$$+ \int_g^\infty p(y)w(-y/2)\varphi(y-g)\,\mathrm{d}y + \int_0^\infty p(y)w(-y/2)\varphi(y+g)\,\mathrm{d}y,$$

where the overdot denotes time derivative. The first term on the right-hand side accounts for the case when the gap length is g and one of the two particles jumps (Fig. 1a and 1b). The second term accounts for the case when a gap of length g is created from a previous gap of length $y < g$ due to the leader jumping forward (Fig. 1c). The third term accounts for the case when a gap of length g is created from a previous gap of length $y > g$ due to the laggard jumping forward, but still remaining the laggard (Fig. 1d). The fourth term accounts for the case when a gap of length g is created from a previous gap of length $y > 0$ due to the laggard jumping forward and overtaking the leader (Fig. 1e). It is easy to check that this conserves the total probability, i.e. $\int_0^\infty \dot{p}(g)\,\mathrm{d}g = 0$.

Our goal is to compute the stationary density, i.e. p such that $\dot{p}(g) = 0$. It is unlikely to be able to find a closed formula for the stationary density in full generality (for general φ and w), and so we restrict ourselves to the special case when $\varphi(x) = \mathrm{e}^{-x}\mathbf{1}[x \ge 0]$ and $w(x) = \mathrm{e}^{-\beta x}$, where $\beta > 0$.

In this case it can be shown (see [3], Section 4.2 for a detailed calculation) that the stationary density of the gap is

$$p(g) = c\frac{1}{(\cosh(\beta g/2))^{1+2/\beta}} \quad \text{for } g > 0,$$

with c being the normalizing constant. In particular, in the special case of $\beta = 2$, the stationary density is

$$p(g) = \frac{1}{\cosh^2(g)} \quad \text{for } g > 0.$$

4.3. *Many-particle case*

In the $n \ge 3$ case with deterministic jumps, the particles $x_1(t), \ldots, x_n(t)$ form a continuous-time Markov process on an n dimensional lattice. Viewed from the mean position of the particles, this becomes a Markov process on an $n-1$ dimensional lattice. However, even in the $n = 3$ case the combinatorics of the problem becomes too difficult to find a closed form for the stationary distribution (see Fig. 2). The model with the jump length distribution having a density presents similar difficulties in the $n \ge 3$ case. (As we have seen, the problem is not completely solved even in the $n = 2$ case.) Understanding the finite particle case is the subject of future work.

5. Mean field

We now turn our attention to the mean field approximation of the model and study the solutions of the mean field equation (3.1). First of all, using the fact that φ is a probability density function, we can check that the mean field equation (3.1) conserves the total probability, i.e. $\int_{-\infty}^\infty \frac{\partial \varrho(x,t)}{\partial t}\,\mathrm{d}x = 0$.
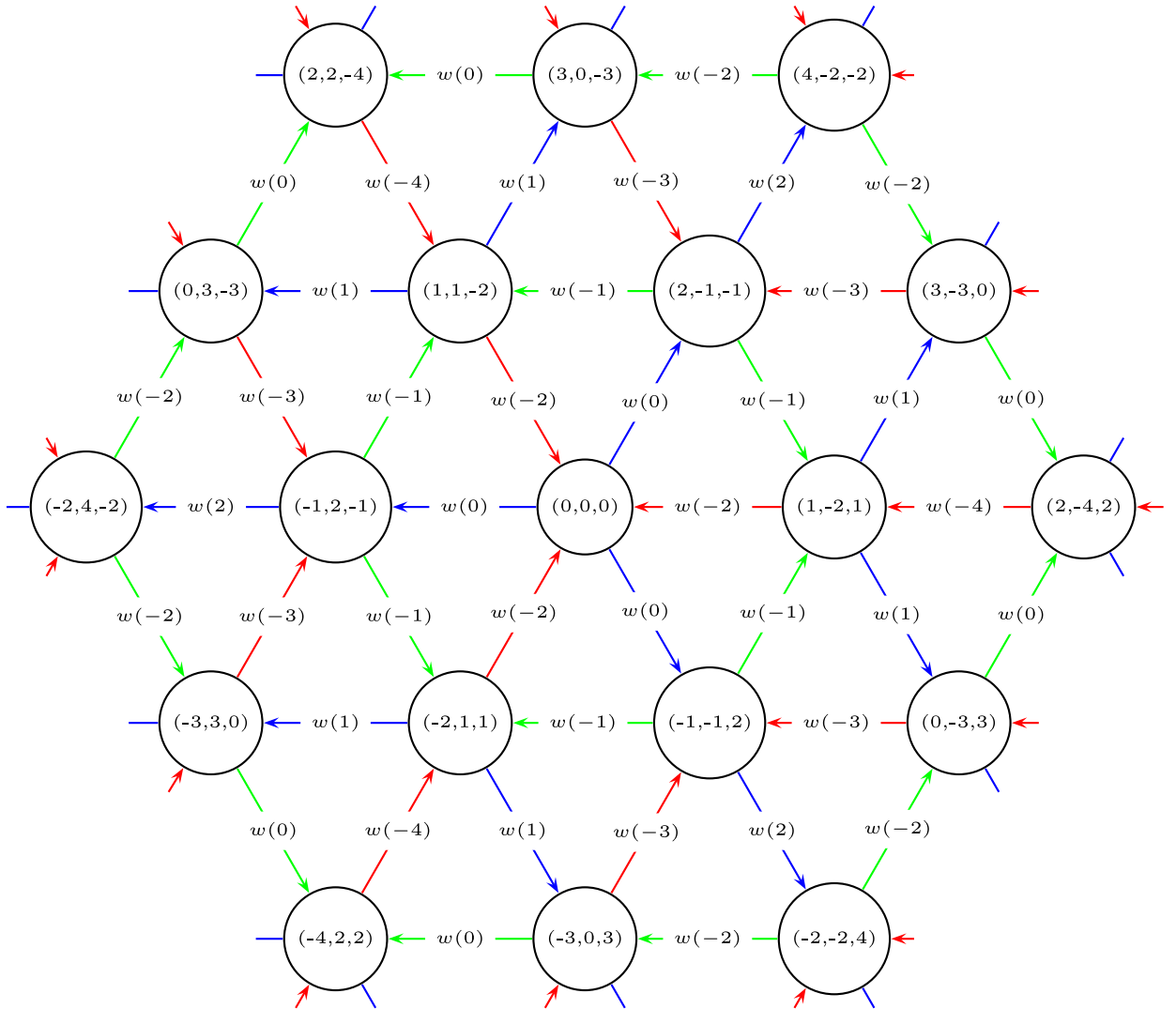
Fig. 2. In the $n = 3$ case with deterministic jumps the particles viewed from the mean position form a continuous-time Markov process on a 2 dimensional lattice with directed edges, as seen above. The coordinates in the nodes show $(3(x_1(t) - m_n(t)), 3(x_2(t) - m_n(t)), 3(x_3(t) - m_n(t)))$. (We multiply by 3 only so that we need not write fractions in the figure.) The rate of going from one node to another on a particular edge is indicated on the edge itself, however, once again the argument of the jump rate function $w(\cdot)$ is multiplied by 3 so that we do not need to write fractions in the figure. The edges going outward from the origin are colored in blue, the ones which are going inward toward the origin are colored in red, while the remaining edges are colored green. Since the rates on the incoming red arrows are larger than those on the outgoing blue arrows, one can see that there is a drift towards the origin, and therefore one expects there to be a stationary distribution on this lattice. However, there can be no detailed balance since there are directed cycles in this graph, and therefore calculating the stationary distribution becomes difficult.

An important quantity is the speed of the mean of the density. This speed can be expressed as a simple integral by differentiating equation (3.2) with respect to time and using the mean field equation (3.1):

$$\dot{m}(t) = \int_{-\infty}^{\infty} x \frac{\partial \varrho(x, t)}{\partial t}\, \mathrm{d}x$$

$$= -\int_{-\infty}^{\infty} x w(x - m(t)) \varrho(x, t)\, \mathrm{d}x + \int_{-\infty}^{\infty} \int_{-\infty}^{x} x w(y - m(t)) \varrho(y, t) \varphi(x - y)\, \mathrm{d}y\, \mathrm{d}x$$

$$= -\int_{-\infty}^{\infty} x w\big(x - m(t)\big)\varrho(x, t)\,\mathrm{d}x + \int_{-\infty}^{\infty} w\big(y - m(t)\big)\varrho(y, t) \int_{y}^{\infty} x\varphi(x - y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= -\int_{-\infty}^{\infty} x w\big(x - m(t)\big)\varrho(x, t)\,\mathrm{d}x + \int_{-\infty}^{\infty} w\big(y - m(t)\big)\varrho(y, t)\,\mathrm{d}y + \int_{-\infty}^{\infty} y w\big(y - m(t)\big)\varrho(y, t)\,\mathrm{d}y$$

$$= \int_{-\infty}^{\infty} w\big(y - m(t)\big)\varrho(y, t)\,\mathrm{d}y, \tag{5.1}$$

where we used that $\int_0^{\infty} x\varphi(x)\,\mathrm{d}x = 1$, as assumed in the model. When $m(t) = ct$ and $\varrho(y, t) = \rho(y - ct)$, equation (5.1) becomes

$$c = \int_{-\infty}^{\infty} w(x)\rho(x)\,\mathrm{d}x. \tag{5.2}$$

We now turn to proving Theorem 3.1.

Proof of Theorem 3.1. Substituting $\varphi(x) = \mathrm{e}^{-x}\mathbf{1}[x \geq 0]$ into (3.4) we get

$$-c\rho'(x) = -w(x)\rho(x) + \int_{-\infty}^{x} w(y)\rho(y)\mathrm{e}^{-(x-y)}\,\mathrm{d}y. \tag{5.3}$$

First, we take the derivative of the integral term in (5.3). Using the property of the exponential function, that its derivative is itself, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-\infty}^{x} w(y)\rho(y)\mathrm{e}^{-(x-y)}\,\mathrm{d}y\right) = w(x)\rho(x) - \int_{-\infty}^{x} w(y)\rho(y)\mathrm{e}^{-(x-y)}\,\mathrm{d}y$$

$$= w(x)\rho(x) - w(x)\rho(x) + c\rho'(x) = c\rho'(x),$$

where we used (5.3) in the second line. It follows that for some constant D we have

$$\int_{-\infty}^{x} w(y)\rho(y)\mathrm{e}^{-(x-y)}\,\mathrm{d}y = c\rho(x) + D.$$

Plugging this into (5.3) we arrive at

$$-c\rho'(x) + w(x)\rho(x) = c\rho(x) + D. \tag{5.4}$$

To show that $D = 0$, let us integrate both sides of the equation with respect to x from $-\infty$ to ∞ (by integrating from $-R$ to R and taking the limit as $R \to \infty$). The integral of the first term on the left-hand side is 0 since the total probability is conserved by the mean field equation, and the integral of the second term on the left-hand side is c according to (5.2). Since ρ is a density we also have that the integral of the first term on the right-hand side is c as well, therefore

$$\lim_{R \to \infty} \int_{-R}^{R} D\,\mathrm{d}x = 0,$$

and thus $D = 0$. Now rearranging (5.4) we get

$$\rho'(x) = \left(\frac{1}{c}w(x) - 1\right)\rho(x)$$

from which we arrive at (3.6). $\qquad\square$

We refer to [3], Section 5 for an alternative proof of Theorem 3.1 by I. P. Tóth.

5.1. *Specific examples*

In this subsection we look at the stationary distribution ρ of (3.6) in the case of two specific jump rate functions. In both cases we also determine the speed of the traveling wave from Theorem 3.1. Details of calculations are given only in the case of Example 5.1, the calculation in the other example is quite simple from Theorem 3.1.

Example 5.1 (Exponential jump rate function). *If $w(x) = e^{-\beta x}$, where $\beta > 0$, then from Theorem 3.1 we obtain*

$$\rho_\beta(x) = K e^{-x - e^{-\beta(x + \log(c\beta)/\beta)}},$$

where K is an appropriate normalizing constant. Integrating this over $\mathbb{R}$ gives us the value for K and thus the density becomes

$$\rho_\beta(x) = \frac{\beta}{\Gamma(1/\beta)} e^{-(x + \log(c\beta)/\beta) - e^{-\beta(x + \log(c\beta)/\beta)}}.$$

To check this let us calculate the integral:

$$\int_{-\infty}^{\infty} \rho_\beta(x)\, dx = \int_{-\infty}^{\infty} \frac{\beta}{\Gamma(1/\beta)} e^{-(x + \log(c\beta)/\beta) - e^{-\beta(x + \log(c\beta)/\beta)}}\, dx$$

$$= \frac{\beta}{\Gamma(1/\beta)} \int_{-\infty}^{\infty} e^{-y - e^{-\beta y}}\, dy = \frac{\beta}{\Gamma(1/\beta)} \int_{\infty}^{0} z^{1/\beta} e^{-z} \left(-\frac{dz}{\beta z}\right) = \frac{1}{\Gamma(1/\beta)} \int_{0}^{\infty} z^{1/\beta - 1} e^{-z}\, dz = 1,$$

using the substitutions $y = x + \frac{\log(c\beta)}{\beta}$ and $z = e^{-\beta y}$. From Theorem 3.1 the speed of the wave c is determined by the fact that ρ_β is centered, so to compute c we must compute the mean of the density. Using similar substitutions as before we get:

$$\int_{-\infty}^{\infty} x \rho_\beta(x)\, dx = -\frac{\log(c\beta)}{\beta} + \int_{-\infty}^{\infty} \left(x + \frac{\log(c\beta)}{\beta}\right) \rho_\beta(x)\, dx = -\frac{\log(c\beta)}{\beta} + \frac{\beta}{\Gamma(1/\beta)} \int_{-\infty}^{\infty} y e^{-y - e^{-\beta y}}\, dy$$

$$= -\frac{\log(c\beta)}{\beta} + \frac{\beta}{\Gamma(1/\beta)} \int_{\infty}^{0} -\frac{\log(z)}{\beta} z^{1/\beta} e^{-z} \left(-\frac{dz}{\beta z}\right)$$

$$= -\frac{\log(c\beta)}{\beta} - \frac{1}{\beta \Gamma(1/\beta)} \int_{0}^{\infty} \log(z) z^{1/\beta - 1} e^{-z}\, dz = -\frac{1}{\beta} \left(\log(c\beta) + \frac{\Gamma'(1/\beta)}{\Gamma(1/\beta)}\right),$$

which is zero when

$$c = \frac{1}{\beta} e^{-\psi(1/\beta)},$$

where ψ is the digamma function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Therefore the stationary distribution in this case is given by (3.7). Note the exponential tail of ρ_β on one side, and the double-exponential tail on the other side. Specifically, in the case of $\beta = 1$ the stationary distribution is the centered standard Gumbel distribution:

$$\rho_1(x) = e^{-(x + \gamma) - e^{-(x + \gamma)}},$$

where γ is Euler's constant. Moreover, it can be seen that ρ_β is a rescaled generalized Gumbel distribution with parameter $1/\beta$ (Bertin [7], Clusel and Bertin [10], Gumbel [21]). As mentioned in Section 3, the (generalized) Gumbel distribution often arises in extreme value theory, so it is natural to ask whether there is an underlying extremal process in the dynamics. It turns out that there is a connection between the mean field model and extreme value theory when $k = 1/\beta$ is an integer: *this is discussed in Section 5.2.*

Example 5.2 (Jump rate function is a step function). *Let*

$$w(x) = \begin{cases} a & \text{if } x < 0, \\ b & \text{if } 0 \le x, \end{cases}$$

where $a > b > 0$. Then from Theorem 3.1 the solution to (5.3) is

$$\rho_{a,b}(x) = \begin{cases} Ke^{(a/c-1)x} & \text{if } x < 0, \\ Ke^{(b/c-1)x} & \text{if } 0 \le x, \end{cases}$$

where the normalizing constant is

$$K = \frac{(a-c)(c-b)}{c(a-b)}.$$

The speed of the wave follows from Theorem 3.1:

$$c = \frac{a+b}{2},$$

and thus the stationary distribution is

$$\rho_{a,b}(x) = \frac{a-b}{2(a+b)} e^{-(a-b)/(a+b)|x|}, \qquad (5.5)$$

which is a Laplace distribution.

For two further examples, see [3], Section 5.1.

5.2. *Connection with extreme value theory*

In the previous subsection we have seen that when $\varphi(x) = e^{-x}\mathbf{1}[x \ge 0]$ and $w(x) = e^{-\beta x}$ then the (rescaled generalized) Gumbel distribution arises as the stationary distribution of the mean field system. This is interesting, because at first sight there is no extremeness in the model. In this section we discuss the connection between the model and extreme value theory. (Here we occasionally neglect mathematical precision for the sake of clarity of the argument.)

Let us start with the standard Gumbel case, i.e. $\beta = 1$. Recall that according to our observation in the Introduction, we can assume that the center of mass moves with a constant velocity c, and we consider a single mean field particle $X(t)$ that jumps with rate $e^{ct-X(t)}$, so

$$\mathbb{P}\big(X(t) \text{ changes between } t \text{ and } t + dt\big) = e^{ct-X(t)}\, dt + o(dt).$$

When this jump occurs, $X(t)$ increases by a random number drawn from an exponential distribution with parameter 1 (denoted by Exp(1) in the sequel).

How can we view this as an extreme process? The answer to this question was given by Attila Rákos, whose idea is the following. As time goes forward, let us take gradually more and more new independent and identically distributed Exp(1) random variables, and let us look at the distribution of the maximum of these random variables, which we denote by $Y(t)$. Specifically, at time t let there be $N(t) = e^{ct}/c$ random variables altogether, so $Y(t)$ is the maximum of $N(t)$ i.i.d. random variables. There are $N'(t)\, dt = e^{ct}\, dt$ new random variables trying to "break the record" between t and $t + dt$, and thus

$$\begin{aligned} \mathbb{P}\big(Y(t) \text{ changes between } t \text{ and } t + dt\big) &= 1 - \big(1 - e^{-Y(t)}\big)^{N'(t)\, dt} \\ &= N'(t)\, dt\, e^{-Y(t)} + o\big(N'(t)\, dt\, e^{-Y(t)}\big) \\ &= e^{ct-Y(t)}\, dt + o\big(e^{ct-Y(t)}\, dt\big) \end{aligned}$$

and if $Y(t)$ changes, then it jumps according to Exp(1). Here we use the memoryless property of the exponential distribution, i.e. the fact that if T is exponentially distributed then

$$\mathbb{P}(T > s + t | T > s) = \mathbb{P}(T > t).$$

Thus we can conclude that the two processes are basically the same. Furthermore, we know that $Y(t) - \log N(t)$ follows the standard Gumbel distribution for large t (see, for example, Gumbel [21] or the excellent survey by Clusel and Bertin [10]):

$$\lim_{t \to \infty} \mathbb{P}\big(Y(t) - \log N(t) < x\big) = e^{-e^{-x}}$$

and therefore since $\log N(t) = ct - \log c$, it follows that

$$\lim_{t \to \infty} \mathbb{P}\big(Y(t) - ct < x\big) = e^{-e^{-(x+\log c)}}.$$

Thus by the connection we have

$$\lim_{t \to \infty} \mathbb{P}\big(X(t) - ct < x\big) = e^{-e^{-(x+\log c)}}$$

and since the distribution of $X(t) - ct$ is centered, it follows that

$$c = e^{-\gamma},$$

where $\gamma = 0.5772\ldots$ is Euler's constant.

The idea above extends to the general case when $k = 1/\beta$ is a fixed positive integer, with an appropriate definition of $Y(t)$. In the following we use k and $1/\beta$ interchangeably.

We define $Y(t)$ as follows. Once again, as time goes forward, let us take gradually more and more new independent and identically distributed Exp(1) random variables. Specifically, at time t let there be $N(t) = e^{\beta ct}/\beta c$ random variables altogether. Denote by $Y_k(t)$ the kth largest value among these random variables at time t, and define $Y(t) := kY_k(t)$. With this definition, the rest of the calculation is done in a similar way to the standard Gumbel case. For a more detailed calculation, see [3], Section 5.2.

Thus we have rediscovered the results of the previous subsection through extreme value theory arguments.

6. Fluid limit

In this section we prove Theorem 3.3. The section is organized as follows. In order to present and prove our results, we need to introduce notation, terminology, definitions and previously known results. This necessary background is introduced in Section 6.1. Then in Section 6.2 we formulate as separate theorems several intermediate steps that lead to the main result and we prove Theorem 3.3 using these steps. In the following subsections we then provide proofs of the intermediate steps stated in Section 6.2.

6.1. *Preliminaries*

Recall that our goal is to show that – given some assumptions – the path of the empirical measure process $\{\mu_n(\cdot)\}_{n \geq 1}$ *converges weakly* in the Skorohod space $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$ to $\mu(\cdot)$, which solves the mean field equation (3.9). The goal of this subsection is to provide all the necessary preliminaries that we need during the proof.

We present important results from the theory of weak convergence that we use later; in particular, we focus on tightness results in Skorohod spaces. For an excellent introduction to weak convergence, we refer the reader to Billingsley [8]. Many results that we use are taken from Ethier and Kurtz [13], and Jacod and Shiryaev [22] is also a valuable source of information concerning convergence of processes.

As in Theorem 3.3, from now on we assume that the jump rate function w is bounded, and we denote its lowest upper bound by a. To put it another way: $a := \lim_{x \to -\infty} w(x) = \sup_x w(x)$. We also assume that w is differentiable and has a bounded derivative. Furthermore, in the following Z always denotes a random variable from the jump length distribution. As before, we assume that $\mathbb{E}(Z) = 1$ and that Z has a finite third moment.

6.1.1. *Probability metrics*

Recall the definition (3.10) of the 1-Wasserstein metric. The following lemma (from Feng and Kurtz [14], Appendix D) summarizes what we need to know about the metric space $(\mathcal{P}_1(\mathbb{R}), d_1)$. More can be found in e.g. Feng and Kurtz [14], Appendix D.

Lemma 6.1. $(\mathcal{P}_1(\mathbb{R}), d_1)$ *is a complete and separable metric space. For $\mu_n, \mu \in \mathcal{P}_1(\mathbb{R})$, the following are equivalent*:

(1) $\lim_{n \to \infty} d_1(\mu_n, \mu) = 0$.
(2) *For all continuous functions φ on $\mathbb{R}$ satisfying $|\varphi(x)| \leq C(1 + |x|)$ for some $C > 0$,*

$$\lim_{n \to \infty} \langle \varphi, \mu_n \rangle = \langle \varphi, \mu \rangle.$$

We introduce another probability metric that we use later on. For $P, Q \in \mathcal{P}_1(\mathbb{R})$ and a set $\mathcal{H}$ of functions define

$$d_{\mathcal{H}}(P, Q) = \sup_{f \in \mathcal{H}} \left| \langle f, P \rangle - \langle f, Q \rangle \right|. \tag{6.1}$$

It is immediate that $d_{\mathcal{H}}$ is a pseudometric on $\mathcal{P}_1(\mathbb{R})$: $d_{\mathcal{H}}(P, P) = 0$, $d_{\mathcal{H}}(P, Q) = d_{\mathcal{H}}(Q, P)$, and the triangle inequality also holds. The only thing that does not necessarily hold is that $d_{\mathcal{H}}(P, Q) = 0$ implies $P = Q$. In order for this to hold, $\mathcal{H}$ has to be a large enough set of functions to be able to distinguish between different probability measures. There are various choices of $\mathcal{H}$ that give rise to notable metrics. For instance, if $\mathcal{H}$ is the set of indicator functions of measurable sets, then $d_{\mathcal{H}}$ is the total variation metric. If $\mathcal{H}$ is the set of half-line indicator functions then $d_{\mathcal{H}}$ is the Kolmogorov (uniform) metric. The choice of $\mathcal{H} = \{f \in C_b : |f| \leq 1\}$ gives the Radon metric (that this is a metric follows from e.g. Billingsley [8], Theorem 1.2.). Consequently the choice $\mathcal{H} = H$ (see (2.1)) also yields a metric. This is what we use in the following: d_H denotes the metric on $\mathcal{P}_1(\mathbb{R})$ given by (6.1) with $\mathcal{H} = H$. This metric is used in Theorem 6.14, Corollary 6.15 and their proofs in Section 6.5. For more on probability metrics we refer the reader to Gibbs and Su [17] for an excellent review.

6.1.2. *Skorohod spaces*

Let (E, r) be a general metric space. Denote by $C([0, \infty), E)$ the space of continuous E-valued paths with the uniform topology. The *Skorohod space* $D([0, \infty), E)$ consists of E-valued paths that are right-continuous and have left limits. We do not go into details about the topology on $D([0, \infty), E)$, but we note that there exists a metric under which $D([0, \infty), E)$ is a complete and separable metric space if (E, r) is complete and separable (Ethier and Kurtz [13], Chapter 3). The topology generated by this metric is called the Skorohod topology on $D([0, \infty), E)$. Note that $C([0, \infty), E)$ is a subset of $D([0, \infty), E)$ and it can be seen that the Skorohod topology relativized to $C([0, \infty), E)$ coincides with the uniform topology there. For more on the Skorohod space we refer the reader to Billingsley [8], Chapter 3 and Ethier and Kurtz [13], Chapter 3.

The reason we introduced the Skorohod space in general is because the natural space in which to consider $\mu_n(\cdot)$ is $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$: we know that $\mu_n(\cdot)$ is a piece-wise constant function in $\mathcal{P}_1(\mathbb{R})$, having jumps whenever a particle jumps. (We can define $\mu_n(\cdot)$ to be right-continuous at its discontinuities.) Since $\mathcal{P}_1(\mathbb{R})$ is complete and separable under the Wasserstein metric, $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$ is also complete and separable under an appropriate metric. Similarly, for any test function f (for instance $f \in H$), the natural space in which to consider $\langle f, \mu_n(\cdot) \rangle$ is the Skorohod space $D([0, \infty), \mathbb{R})$.

6.1.3. *Tightness in Skorohod spaces*

Recall our plan of action from the end of Section 3:

- Prove tightness of $\{\mu_n(\cdot)\}_{n \geq 1}$ in the Skorohod space $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$.
- Identify the limit of $\{\mu_n(\cdot)\}_{n \geq 1}$: here we essentially have to show that the limit solves the deterministic mean field equation.
- Prove uniqueness of the solution to the mean field equation with a given initial condition.

In view of the first point, our goal here is to find checkable sufficient conditions for tightness in Skorohod spaces. Since all metric spaces we deal with are complete and separable, by Prohorov's theorem the concepts of tightness and

relative compactness are equivalent in our setting, and therefore we use these two notions interchangeably. Following Perkins [26], we say that a sequence is *C*-relatively compact if it is relatively compact and all weak limit points are a.s. continuous.

Tightness in $D([0, \infty), \mathbb{R})$.

First, we state conditions for tightness in $D([0, \infty), \mathbb{R})$. For $y(\cdot) \in D([0, \infty), \mathbb{R})$ define

$$J_T(y(\cdot)) = \sup_{0 \leq t \leq T} |y(t) - y(t-)|, \tag{6.2}$$

the largest (absolute) jump in $y(\cdot)$ until T. Let $\{Y_n(\cdot)\}_{n \geq 1}$ be a sequence of random variables in $D([0, \infty), \mathbb{R})$. The following theorem, which is a combination of Billingsley [8], Theorem 16.8. and the corollary thereafter, characterizes tightness in $D([0, \infty), \mathbb{R})$.

Theorem 6.2. *The sequence $\{Y_n(\cdot)\}_{n \geq 1}$ is tight in $D([0, \infty), \mathbb{R})$ if and only if the following three conditions hold*:

(i) $\lim_{L \to \infty} \limsup_{n \to \infty} \mathbb{P}(|Y_n(0)| \geq L) = 0.$ (6.3)

(ii) *For each $T \geq 0$,*

$$\lim_{L \to \infty} \limsup_{n \to \infty} \mathbb{P}(J_T(Y_n(\cdot)) \geq L) = 0. \tag{6.4}$$

(iii) *For each $T \geq 0$ and $\varepsilon > 0$,*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}\Big(\inf_{\{t_i\}} \max_{1 \leq i \leq v} \sup_{s,t \in [t_{i-1}, t_i)} |Y_n(t) - Y_n(s)| \geq \varepsilon\Big) = 0, \tag{6.5}$$

where the infimum extends over all decompositions $[t_{i-1}, t_i), 1 \leq i \leq v$, of $[0, T)$ such that $t_i - t_{i-1} > \delta$ for $1 \leq i < v$.

We now look at a condition which implies that any limit point is continuous, i.e. is in $C([0, \infty), \mathbb{R})$. Following Ethier and Kurtz [13], p. 147, for $y(\cdot) \in D([0, \infty), \mathbb{R})$ define

$$J(y(\cdot)) = \int_0^\infty e^{-u} \min\{J_u(y(\cdot)), 1\} \, du, \tag{6.6}$$

where $J_u(y(\cdot))$ is defined as in (6.2). The following theorem characterizes continuity of a weak limit point (Ethier and Kurtz [13], Theorem 3.10.2.).

Theorem 6.3. *Suppose $Y_n(\cdot) \Rightarrow_n Y(\cdot)$ in $D([0, \infty), \mathbb{R})$. Then $Y(\cdot)$ is almost surely continuous if and only if $J(Y_n(\cdot)) \Rightarrow_n 0$.*

We now give sufficient conditions for a sequence to be *C*-relatively compact.

Theorem 6.4. *Suppose the following two conditions hold.*

(i)

$$\lim_{L \to \infty} \limsup_{n \to \infty} \mathbb{P}(|Y_n(0)| \geq L) = 0. \tag{6.7}$$

(ii) *For each $T \geq 0$ and $\varepsilon > 0$,*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}\Big(\sup_{\substack{0 < t-s < \delta \\ 0 \leq s \leq t \leq T}} |Y_n(t) - Y_n(s)| \geq \varepsilon\Big) = 0. \tag{6.8}$$

Then $\{Y_n(\cdot)\}_{n \geq 1}$ is C-relatively compact in $D([0, \infty), \mathbb{R})$.

This theorem is stated in Grigorescu and Kang [20], Section 3, but without a proof, and so for completeness we provide a short proof.

Proof. Let us check tightness first: according to Theorem 6.2 we need to check that (6.3), (6.4), and (6.5) hold. Our first condition is exactly (6.3). From (6.8) it follows that for every $T > 0$, $J_T(Y_n(\cdot)) \Rightarrow_n 0$, and consequently (6.4) holds. To show that (6.5) follows from (6.8), note that

$$\inf_{\{t_i\}} \max_{1 \leq i \leq v} \sup_{s,t \in [t_{i-1}, t_i)} \left| Y_n(t) - Y_n(s) \right| \leq \sup_{\substack{0 < t-s < 2\delta \\ 0 \leq s \leq t \leq T}} \left| Y_n(t) - Y_n(s) \right|,$$

where the infimum extends over the same range as in Theorem 6.2.

Now let us check that any weak limit point is a.s. continuous. According to Theorem 6.3, we need to check that $J(Y_n(\cdot)) \Rightarrow_n 0$. From (6.8) we know that for every $T > 0$, $J_T(Y_n(\cdot)) \Rightarrow_n 0$. For any fixed $\varepsilon > 0$ there exists a T such that $\mathrm{e}^{-T} < \varepsilon/2$. From (6.6) it follows that $J(Y_n(\cdot)) \leq T J_T(Y_n(\cdot)) + \mathrm{e}^{-T}$, and consequently:

$$\lim_{n \to \infty} \mathbb{P}\left(J\left(Y_n(\cdot)\right) > \varepsilon\right) \leq \lim_{n \to \infty} \mathbb{P}\left(J_T\left(Y_n(\cdot)\right) > \varepsilon/2T\right) = 0. \qquad \Box$$

Tightness in Skorohod spaces containing measure-valued paths

Now let us turn to conditions that imply tightness in the Skorohod space of time-indexed measure-valued paths $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$. Our main tool is a slight modification of a theorem of Perkins [26], Theorem II.4.1. Perkins' theorem basically states that if we can check an extra condition, then C-relative compactness in $D([0, \infty), M_F(E))$ (where $M_F(E)$ is the space of finite measures on the metric space E, with the topology of weak convergence) can be checked by checking the C-relative compactness of appropriate integrals in $D([0, \infty), \mathbb{R})$.

Our modification of Perkins' theorem is the following.

Theorem 6.5. *A sequence of cadlag $\mathcal{P}_1(\mathbb{R})$-valued processes $\{P_n(\cdot)\}_{n \geq 1}$ is C-relatively compact in $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$ if the following conditions hold*:

(i) *For every $T > 0$ there exists a $c = c(T)$ and a $\tau = \tau(T) \in (0, 1)$ such that for every $\varepsilon > 0$ there exists a $K = K_{T,\varepsilon} \in \mathbb{R}$ such that $K \leq c/\varepsilon^{1-\tau}$ and*

$$\sup_n \mathbb{P}\left(\sup_{0 \leq t \leq T} P_n\left(t, (-\infty, -K) \cup (K, \infty)\right) > \varepsilon \right) < \varepsilon.$$

(ii) *For every $f \in C_b$, $\{\langle f, P_n(\cdot) \rangle\}_{n \geq 1}$ is C-relatively compact in $D([0, \infty), \mathbb{R})$.*

If one looks at the proof of Perkins' original theorem [26], Theorem II.4.1, then one can see that it holds true if one replaces $M_F(E)$ with $M_1(E)$, the space of probability measures on E with the topology of weak convergence. However, our measure-valued process lives in a different space, namely $\mu_n(\cdot) \in D([0, \infty), \mathcal{P}_1(\mathbb{R}))$, where the Wasserstein space $\mathcal{P}_1(\mathbb{R})$ is endowed with the metric d_1. It turns out that we cannot simply replace $M_F(E)$ with $\mathcal{P}_1(\mathbb{R})$ in the theorem. Convergence in $(\mathcal{P}_1(\mathbb{R}), d_1)$ is stronger than weak convergence: it is weak convergence and the convergence of the first absolute moment of the measure. It is this that causes the original proof of the theorem to break down.

The proof of Theorem 6.5 is essentially the same as that of Perkins' theorem, but with a slight modification at one point, which requires that our condition (i) is stronger than in Perkins' original version. For completeness, we reproduce the whole proof, following the lines of Perkins [26], Section II.4.: this can be found in Section 6.6.

We use Theorem 6.5 in order to prove tightness of the empirical measure process $\{\mu_n(\cdot)\}_{n \geq 1}$ in Corollary 6.10.

6.1.4. *A few more facts that are used*

Before we move on to our results, let us finish the preliminaries with a few more things that we use later in the proofs (in particular in Section 6.4). We rely on the continuous mapping theorem (Billingsley [8], Section 2). We do not need the theorem in full generality, only the following:

Theorem 6.6. *Suppose h is a continuous function from the metric space (S, d) to the metric space (S', d'). If $X_n \Rightarrow_n X$ in (S, d), then $h(X_n) \Rightarrow_n h(X)$ in (S', d').*

Finally, we need the following two lemmas, which can be found as exercises in Ethier and Kurtz [13], Chapter 3.

Lemma 6.7. *Let E and F be metric spaces, and let $f : E \to F$ be continuous. Then the mapping $x \to f \circ x$ from $D([0, \infty), E)$ to $D([0, \infty), F)$ is continuous.*

Lemma 6.8. *The function*

$$f(x)(\cdot) = \int_0^\cdot x(s) \, ds$$

from $D([0, \infty), \mathbb{R})$ to $D([0, \infty), \mathbb{R})$ is continuous.

6.2. *Main steps of the fluid limit procedure*

In this subsection we collect the intermediate steps that build up the proof of the main theorem, Theorem 3.3, leaving the proofs of these results to later subsections. We complete the proof of Theorem 3.3 at the end of the subsection. Recall our assumptions on the jump rate function w and the jump length Z which we assume throughout this section.

Our first results deal with the tightness of $\{\mu_n(\cdot)\}_{n \geq 1}$ in the Skorohod space $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$.

Theorem 6.9. *Suppose Assumption 1 from Section 3 holds. Then for any $f \in C_b$ the sequence $\{\langle f, \mu_n(\cdot) \rangle\}_{n \geq 1}$ is C-relatively compact in $D([0, \infty), \mathbb{R})$.*

We remark that although we only need the above result for continuous and bounded functions in the corollary below, continuity is not used in our proof, and with a slight modification in the proof the same can be proven for Lipschitz continuous functions.

Corollary 6.10. *Suppose Assumptions 1 and 2 from Section 3 hold. Then $\{\mu_n(\cdot)\}_{n \geq 1}$ is C-relatively compact in $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$.*

Theorem 6.9 and Corollary 6.10 are proved in Section 6.3.

We remark that tightness can be proved in other ways as well, using the techniques of Ethier and Kurtz [13], Chapter 3. In particular, Thomas G. Kurtz communicated to us a different way of checking tightness, and we have included a sketch of his proof at the end of Section 6.3.

Our next group of results is concerned with identifying the limit of the sequence $\{\mu_n(\cdot)\}_{n \geq 1}$. In particular, our goal is to show that any weak limit point of $\{\mu_n(\cdot)\}_{n \geq 1}$ solves the mean field equation. We prove this in two steps. Recall the definition (2.1) of H.

Theorem 6.11. *For every $t \geq 0$ and every $f \in H$,*

$$\sup_{0 \leq s \leq t} \left| A_{s,f}\big(\mu_n(\cdot)\big) \right| \xrightarrow[n \to \infty]{\mathbb{P}} 0, \tag{6.9}$$

where $\xrightarrow[n \to \infty]{\mathbb{P}} 0$ denotes convergence in probability.

Theorem 6.12. *If $\mu_n(\cdot) \Rightarrow_n \mu(\cdot)$ in $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$, then for every $t \geq 0$ and every $f \in H$,*

$$A_{t,f}\big(\mu_n(\cdot)\big) \Rightarrow_n A_{t,f}\big(\mu(\cdot)\big) \tag{6.10}$$

in $\mathbb{R}$.

An immediate corollary of these two theorems is the following.

Corollary 6.13. *Any weak limit point $\mu(\cdot)$ of $\{\mu_n(\cdot)\}_{n\geq 1}$ in the Skorohod space $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$ solves the mean field equation almost surely. That is, if $\mu_n(\cdot) \Rightarrow_n \mu(\cdot)$ in $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$, then for every $t \geq 0$ and every $f \in H$, $A_{t,f}(\mu(\cdot)) = 0$ almost surely.*

The proofs of Theorem 6.11, Theorem 6.12, and Corollary 6.13 can be found in Section 6.4.

Our final result concerns the mean field equation: the solution with a given initial condition is unique.

Theorem 6.14. *Suppose $\mu^1(\cdot)$ and $\mu^2(\cdot)$ are solutions to the mean field equation with initial conditions μ_0^1 and μ_0^2 respectively. Denote by $d_H(t)$ the distance $d_H(\mu^1(t), \mu^2(t))$ of the two measures at time t according to the metric defined by (6.1) with $\mathcal{H} = H$. Then there exists a constant c such that*

$$d_H(t) \leq d_H(0)e^{ct}. \tag{6.11}$$

Corollary 6.15. *Suppose $\mu^1(\cdot)$ and $\mu^2(\cdot)$ are solutions to the mean field equation with the same initial condition. Then $\mu^1(\cdot) = \mu^2(\cdot)$, i.e. the solution to the mean field equation with a given initial condition is unique.*

Let us now prove our main result.

Proof of Theorem 3.3. The sequence $\{\mu_n(\cdot)\}_{n\geq 1}$ is tight in $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$ due to Corollary 6.10. Corollary 6.13 shows that any limit point satisfies the mean field equation. Due to Assumption 3, any limit point satisfies the mean field equation with deterministic initial condition ν. We know that the solution to the mean field equation with a given initial condition is unique (Corollary 6.15). Consequently, due to Prohorov's theorem, $\mu_n(\cdot) \Rightarrow_n \mu(\cdot)$, since we have proved tightness and the uniqueness of any limit point. $\qquad\square$

6.3. *Tightness*

In this subsection we prove Theorem 6.9 and Corollary 6.10.

Before starting the proof, let us introduce a particle system $\{\tilde{x}_i(\cdot)\}_{i=1}^n$, related to $\{x_i(\cdot)\}_{i=1}^n$, which we use in the proof. Initially, the two particle systems have the same configurations: $\tilde{x}_i(0) = x_i(0)$ for each i. Let $\{\tilde{x}_i(\cdot)\}_{i=1}^n$ be a particle system in which all the particles are independent from each other, all of the particles jump with rate a (recall that $a = \lim_{x \to -\infty} w(x) = \sup_x w(x)$), and whenever they jump, the jump length is a random variable from the same jump length distribution of the original system, independently of everything else. In other words, let $S_i(\cdot)$ be independent Poisson processes with rate a for each i, let $\xi_1^i, \xi_2^i, \ldots$ be i.i.d. random variables from the jump length distribution, and define $\tilde{x}_i(t) = x_i(0) + \sum_{k=1}^{S_i(t)} \xi_k^i$. Moreover, $\{x_i(\cdot)\}_{i=1}^n$ and $\{\tilde{x}_i(\cdot)\}_{i=1}^n$ can be constructed jointly on the same probability space in the following way. Define $\{\tilde{x}_i(\cdot)\}_{i=1}^n$ exactly as above, and whenever a particle $\tilde{x}_i$ jumps, let x_i jump the same jump length with probability $w(x_i(t) - m_n(t))/a$; with the remaining probability let x_i stay in place. This joint construction leads to two advantageous monotonicity properties: first,

$$\tilde{x}_i(t) \geq x_i(t)$$

for any i and $t \geq 0$, and second,

$$\tilde{x}_i(t) - \tilde{x}_i(s) \geq x_i(t) - x_i(s) \tag{6.12}$$

for any i and $t \geq s \geq 0$.

Proof of Theorem 6.9. Let $f \in C_b$, and denote by K the upper bound of $|f|$, i.e. $|f(x)| \leq K$ for all x. Define $I_n(\cdot) := \langle f, \mu_n(\cdot) \rangle = \frac{1}{n} \sum_{i=1}^n f(x_i(\cdot))$. Clearly $|I_n(t)| \leq K$ for any t. In order to prove tightness in $D([0, \infty), \mathbb{R})$ we

need to show that (6.7) and (6.8) hold with Y_n replaced with I_n. Since f is bounded it is immediate that (6.7) holds. Now let us show that (6.8) holds. Our basic observation is that

$$\left|I_n(t) - I_n(s)\right| = \frac{1}{n}\left|\sum_{i=1}^{n}\left(f\left(x_i(t)\right) - f\left(x_i(s)\right)\right)\right| \le \frac{1}{n}\sum_{i=1}^{n}\left|f\left(x_i(t)\right) - f\left(x_i(s)\right)\right|$$

$$\le \frac{2K}{n}\sum_{i=1}^{n}\mathbf{1}\left[x_i(t) > x_i(s)\right].$$

By monotonicity of $x_i(\cdot)$ and using (6.12) we have

$$\mathbb{P}\left(\sup_{\substack{0<t-s<\delta\\0\le s\le t\le T}}\left|I_n(t) - I_n(s)\right| > \varepsilon\right) \le \mathbb{P}\left(\sup_{\substack{0<t-s<\delta\\0\le s\le t\le T}}\frac{2K}{n}\sum_{i=1}^{n}\mathbf{1}\left[x_i(t) > x_i(s)\right] > \varepsilon\right)$$

$$\le \mathbb{P}\left(\sup_{0\le s\le T}\frac{2K}{n}\sum_{i=1}^{n}\mathbf{1}\left[x_i(s+2\delta) > x_i(s)\right] > \varepsilon\right)$$

$$\le \mathbb{P}\left(\sup_{0\le s\le T}\frac{2K}{n}\sum_{i=1}^{n}\mathbf{1}\left[\tilde{x}_i(s+2\delta) > \tilde{x}_i(s)\right] > \varepsilon\right).$$

To abbreviate notation, define $Y_{n,\delta}(s) := \frac{2K}{n}\sum_{i=1}^{n}\mathbf{1}[\tilde{x}_i(s+\delta) > \tilde{x}_i(s)]$. Now if $\sup_{0\le s\le T} Y_{n,2\delta}(s) > \varepsilon$, then there exists an integer k, $0 \le k \le \lceil T/\delta\rceil - 1$, such that $Y_{n,\delta}(k\delta) \ge \varepsilon/4$. Suppose the contrary, that for every k the above expression is less than $\varepsilon/4$; then since an interval of length 2δ can intersect at most three neighboring intervals of length δ, it follows that $\sup_{0\le s\le T} Y_{n,2\delta}(s) < 3\varepsilon/4$. Therefore

$$\mathbb{P}\left(\sup_{0\le s\le T} Y_{n,2\delta}(s) > \varepsilon\right) \le \mathbb{P}\left(\exists k, 0 \le k \le \lceil T/\delta\rceil - 1: Y_{n,\delta}(k\delta) \ge \varepsilon/4\right)$$

$$= 1 - \mathbb{P}\left(\forall k, 0 \le k \le \lceil T/\delta\rceil - 1: Y_{n,\delta}(k\delta) < \varepsilon/4\right)$$

$$= 1 - \left(\mathbb{P}\left(Y_{n,\delta}(0) < \varepsilon/4\right)\right)^{\lceil T/\delta\rceil}$$

$$= 1 - \left(1 - \mathbb{P}\left(Y_{n,\delta}(0) \ge \epsilon/4\right)\right)^{\lceil T/\delta\rceil},$$

where we used that for every i, $\tilde{x}_i(\cdot)$ has independent and stationary increments. We bound the probability in the last line using Markov's inequality:

$$\mathbb{P}\left(\sum_{i=1}^{n}\mathbf{1}\left[\tilde{x}_i(\delta) > \tilde{x}_i(0)\right] > \frac{n\varepsilon}{8K}\right) \le \mathbb{E}\left(\left(\sum_{i=1}^{n}\mathbf{1}\left[\tilde{x}_i(\delta) > \tilde{x}_i(0)\right]\right)^2\right)\bigg/\frac{n^2\varepsilon^2}{64K^2},$$

where

$$\mathbb{E}\left(\left(\sum_{i=1}^{n}\mathbf{1}\left[\tilde{x}_i(\delta) > \tilde{x}_i(0)\right]\right)^2\right) = n\mathbb{P}\left(\tilde{x}_1(\delta) > \tilde{x}_1(0)\right) + \left(n^2 - n\right)\mathbb{P}\left(\tilde{x}_1(\delta) > \tilde{x}_1(0) \text{ and } \tilde{x}_2(\delta) > \tilde{x}_2(0)\right)$$

$$= n\left(1 - e^{-a\delta}\right) + \left(n^2 - n\right)\left(1 - e^{-a\delta}\right)^2$$

$$\approx na\delta + \left(n^2 - n\right)(a\delta)^2,$$

from which (6.8) follows immediately. □

Proof of Corollary 6.10. In order to prove this corollary we need to check the two conditions of Theorem 6.5. The second condition follows immediately from Theorem 6.9. It remains to show the first condition of Theorem 6.5. Due to monotonicity of the $x_i(t)$ in t, we have

$$\sup_{0 \le t \le T} \mu_n\big(t, (-\infty, -K) \cup (K, \infty)\big) \le \sup_{0 \le t \le T} \mu_n\big(t, (-\infty, -K)\big) + \sup_{0 \le t \le T} \mu_n\big(t, (K, \infty)\big)$$

$$= \mu_n\big(0, (-\infty, -K)\big) + \mu_n\big(T, (K, \infty)\big)$$

and consequently

$$\sup_n \mathbb{P}\Big(\sup_{0 \le t \le T} \mu_n\big(t, (-\infty, -K) \cup (K, \infty)\big) > \varepsilon \Big)$$

$$\le \sup_n \mathbb{P}\big(\mu_n\big(0, (-\infty, -K)\big) + \mu_n\big(T, (K, \infty)\big) > \varepsilon\big)$$

$$\le \sup_n \mathbb{P}\big(\mu_n\big(0, (-\infty, -K)\big) > \varepsilon/2\big) + \sup_n \mathbb{P}\big(\mu_n\big(T, (K, \infty)\big) > \varepsilon/2\big).$$

Let us look at the second term:

$$\mu_n\big(T, (K, \infty)\big) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\big[x_i(T) > K\big]$$

$$\le \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\big[x_i(0) > K/2\big] + \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\big[x_i(T) - x_i(0) > K/2\big]$$

$$\le \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\big[x_i(0) > K/2\big] + \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\big[\tilde{x}_i(T) - \tilde{x}_i(0) > K/2\big],$$

and so consequently

$$\sup_n \mathbb{P}\big(\mu_n\big(T, (K, \infty)\big) > \varepsilon/2\big) \le \sup_n \mathbb{P}\big(\mu_n\big(0, (K/2, \infty)\big) > \varepsilon/4\big)$$

$$+ \sup_n \mathbb{P}\bigg(\frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\big[\tilde{x}_i(T) - \tilde{x}_i(0) > K/2\big] > \varepsilon/4\bigg).$$

Let us look at the second term on the right-hand side. Using Markov's inequality and the fact that the $\tilde{x}_i$'s are identically distributed, we have

$$\mathbb{P}\bigg(\frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\big[\tilde{x}_i(T) - \tilde{x}_i(0) > K/2\big] > \varepsilon/4\bigg) \le \frac{4}{\varepsilon} \mathbb{P}\big(\tilde{x}_1(T) - \tilde{x}_1(0) > K/2\big).$$

Now using Markov's inequality again, we get that

$$\frac{4}{\varepsilon} \mathbb{P}\big(\tilde{x}_1(T) - \tilde{x}_1(0) > K/2\big) = \frac{4}{\varepsilon} \mathbb{P}\big(\big(\tilde{x}_1(T) - \tilde{x}_1(0)\big)^3 > K^3/8\big) \le \frac{32}{K^3 \varepsilon} \mathbb{E}\big(\big(\tilde{x}_1(T) - \tilde{x}_1(0)\big)^3\big) =: \frac{\tilde{c}(T)}{K^3 \varepsilon}.$$

Now putting everything together we get

$$\sup_n \mathbb{P}\Big(\sup_{0 \le t \le T} \mu_n\big(t, (-\infty, -K) \cup (K, \infty)\big) > \varepsilon \Big) \le \sup_n \mathbb{P}\big(\mu_n\big(0, (-\infty, -K)\big) > \varepsilon/2\big)$$

$$+ \sup_n \mathbb{P}\big(\mu_n\big(0, (K/2, \infty)\big) > \varepsilon/4\big) + \frac{\tilde{c}(T)}{K^3 \varepsilon}$$

and we need the left-hand side to be at most ε, for which it is enough that the right-hand side is at most ε.

So let $T > 0$ and $\varepsilon > 0$ be fixed, and let us choose $c = c(T)$, $\tau = \tau(T)$, and $K = K_{T,\varepsilon}$ as follows:

$$c(T) := \max\{3\tilde{c}(T), 8\hat{c}^3\},$$

$$\tau(T) := 1/3,$$

$$K_{T,\varepsilon} := c(T)^{1/3}\varepsilon^{-2/3},$$

where $\hat{c}$ is given by Assumption 2. Then $K^3 \geq 3\tilde{c}(T)\varepsilon^{-2}$, and so $\tilde{c}(T)/(K^3\varepsilon) \leq \varepsilon/3$. On the other hand $K \geq 2\hat{c}\varepsilon^{-2/3}$, and so using Assumption 2 we have

$$\sup_n \mathbb{P}\big(\mu_n\big(0, (-\infty, -K)\big) > \varepsilon/2\big) \leq \varepsilon/3,$$

$$\sup_n \mathbb{P}\big(\mu_n\big(0, (K/2, \infty)\big) > \varepsilon/4\big) \leq \varepsilon/3.$$

Putting all these together concludes the proof. $\qquad\square$

As mentioned before, there are other ways to check tightness, and here we sketch Thomas G. Kurtz's proof (see also Kotelenez and Kurtz [23], Appendix). First note that

$$d_1(\mu, \nu) \leq \mathbb{E}|X - Y|,$$

where X and Y are any random variables with distributions μ and ν respectively. The paths of our particles $\{x_i(\cdot)\}_{i=1}^n$ are exchangeable, and so for any $\{\mathcal{F}_t^{\mu_n(\cdot)}\}$-stopping time τ we have

$$\mathbb{E}\big(d_1\big(\mu_n(\tau), \mu_n(\tau + u)\big)\big) \leq \mathbb{E}\big(x_1(\tau + u) - x_1(\tau)\big). \tag{6.13}$$

For each $T > 0$, let $S_n(T)$ be the collection of $\{\mathcal{F}_t^{\mu_n(\cdot)}\}$-stopping times τ satisfying $\tau \leq T$. Then by (6.13) and Aldous's criterion (see e.g. Ethier and Kurtz [13], Theorem 8.6), $\{\mu_n(\cdot)\}_{n\geq 1}$ is tight in $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$ if for each $T > 0$,

$$\lim_{u\to 0} \limsup_{n\to\infty} \sup_{\tau\in S_n(T)} \mathbb{E}\big(x_1(\tau + u) - x_1(\tau)\big) = 0$$

and also condition (a) of [13], Theorem 7.2 holds. In our case, for a bounded jump rate function w, these conditions can be checked just like before, in the previous way of showing tightness.

6.4. *Identifying the limit*

In this section we prove Theorem 6.11, Theorem 6.12, and Corollary 6.13.

Proof of Theorem 6.11. Let $\mathcal{F}_t$ denote the filtration generated by the n particles at time t. If $X(t)$ is measurable with respect to $\mathcal{F}_t$ then let

$$LX(t) := \lim_{dt\to 0} \frac{1}{dt}\mathbb{E}\big(X(t + dt) - X(t)|\mathcal{F}_t\big).$$

We know that $X(t) - X(0) - \int_0^t LX(s)\,ds$ is a martingale. Let $t > 0$ and $f \in H$ be fixed from now on. As in the previous section, define $I_n(\cdot) := \langle f, \mu_n(\cdot)\rangle = \frac{1}{n}\sum_{i=1}^n f(x_i(\cdot))$. Calculation shows that

$$LI_n(t) = \big\langle\big(\mathbb{E}\big(f(x + Z)\big) - f(x)\big)w\big(x - m_n(t)\big), \mu_n(t)\big\rangle$$

which means that $M_n(t) := A_{t,f}(\mu_n(\cdot))$ is a martingale in t with $M_n(0) = 0$. Therefore by Doob's inequality (see e.g. Jacod and Shiryaev [22], p. 11) we have that for any $c > 0$,

$$\mathbb{P}\bigg(\sup_{0\leq s\leq t}\big|M_n(s)\big| \geq c\bigg) \leq \frac{\mathbb{E}(\sup_{0\leq s\leq t} M_n^2(s))}{c^2} \leq \frac{4\mathbb{E}(M_n^2(t))}{c^2}.$$

So in order to show (6.9) we need to show that

$$\lim_{n \to \infty} \mathbb{E}\big(M_n^2(t)\big) = 0. \tag{6.14}$$

We know that

$$N_n(t) := M_n^2(t) - \int_0^t LM_n^2(s)\, \mathrm{d}s$$

is a martingale with $N_n(0) = 0$, and therefore

$$\mathbb{E}\big(M_n^2(t)\big) = \mathbb{E}\bigg(\int_0^t LM_n^2(s)\, \mathrm{d}s\bigg) = \int_0^t \mathbb{E}\big(LM_n^2(s)\big)\, \mathrm{d}s. \tag{6.15}$$

The order of the expectation and the integral can be switched due to the Fubini–Tonelli theorem, because $LM_n^2(s) \geq 0$ as we will see shortly. Let us calculate $LM_n^2(t)$:

$$LM_n^2(t) = \lim_{\mathrm{d}t \to 0} \frac{1}{\mathrm{d}t} \mathbb{E}\big(M_n^2(t + \mathrm{d}t) - M_n^2(t) | \mathcal{F}_t\big)$$

$$= \lim_{\mathrm{d}t \to 0} \frac{1}{\mathrm{d}t} \mathbb{E}\big(\big(M_n(t + \mathrm{d}t) - M_n(t)\big)^2 | \mathcal{F}_t\big),$$

where we used that $M_n(t)$ is a martingale. Since $LI_n(s)$ is bounded, we have

$$M_n(t + \mathrm{d}t) - M_n(t) = \big(I_n(t + \mathrm{d}t) - I_n(t)\big) - \int_t^{t+dt} LI_n(s)\, \mathrm{d}s$$

$$= \big(I_n(t + \mathrm{d}t) - I_n(t)\big) + \mathrm{O}(\mathrm{d}t)$$

and so consequently

$$LM_n^2(t) = \lim_{\mathrm{d}t \to 0} \frac{1}{\mathrm{d}t} \mathbb{E}\big(\big(M_n(t + \mathrm{d}t) - M_n(t)\big)^2 | \mathcal{F}_t\big)$$

$$= \lim_{\mathrm{d}t \to 0} \frac{1}{\mathrm{d}t} \mathbb{E}\big(\big(I_n(t + \mathrm{d}t) - I_n(t)\big)^2 | \mathcal{F}_t\big)$$

$$= \frac{1}{n^2} \lim_{\mathrm{d}t \to 0} \frac{1}{\mathrm{d}t} \mathbb{E}\bigg(\bigg(\sum_{i=1}^n \big(f\big(x_i(t + \mathrm{d}t)\big) - f\big(x_i(t)\big)\big)\bigg)^2 \Big| \mathcal{F}_t\bigg)$$

$$= \frac{1}{n^2} \sum_{i=1}^n w\big(x_i(t) - m_n(t)\big) \mathbb{E}\big(\big(f\big(x_i(t) + Z\big) - f\big(x_i(t)\big)\big)^2\big).$$

Now if $f \in C_b$, $|f| \leq 1$, then $LM_n^2(t) \leq 4a/n$. If $f = \mathrm{Id}$, then $LM_n^2(t) \leq aE(Z^2)/n$. In both cases this shows, via (6.15), that (6.14) holds. □

Proof of Theorem 6.12. Recall that $w \leq a$ and that $|w'| \leq a'$.

We first show that for every $t \geq 0$ and every $f \in H$,

$$\langle f, \mu_n(t) \rangle \Rightarrow_n \langle f, \mu(t) \rangle \tag{6.16}$$

in $\mathbb{R}$. First of all, we know that the projection $\pi_t : D([0, \infty), \mathcal{P}_1(\mathbb{R})) \to \mathcal{P}_1(\mathbb{R})$, given by $\pi_t(v(\cdot)) = v(t)$, is continuous at $v(\cdot)$ if and only if $v(\cdot)$ is continuous at t. Since $\mu(\cdot) \in C([0, \infty), \mathcal{P}_1(\mathbb{R}))$ a.s. (by Corollary 6.10), the continuous mapping theorem and the fact that $\mu_n(\cdot) \Rightarrow_n \mu(\cdot)$ in $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$ together imply that for every $t \geq 0$,

$$\mu_n(t) \Rightarrow_n \mu(t)$$

in $\mathcal{P}_1(\mathbb{R})$. Now for any $f \in H$, the function $h_f : \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$, given by $h_f(\nu) = \langle f, \nu \rangle$, is continuous. (This follows easily from the equivalence of (1) and (2) in Lemma 6.1.) Using the continuous mapping theorem, this shows (6.16).

We introduce $g_f(x) := (\mathbb{E}(f(x + Z)) - f(x))$ in order to abbreviate notation. What remains to be shown is that for every $t \geq 0$ and every $f \in H$,

$$\int_0^t \langle g_f(x)w(x - m_n(s)), \mu_n(s) \rangle \mathrm{d}s \Rightarrow_n \int_0^t \langle g_f(x)w(x - m(s)), \mu(s) \rangle \mathrm{d}s \tag{6.17}$$

in $\mathbb{R}$. First, let us show that for any function $f \in H$, the function $h_f : \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$, given by $h_f(\nu) = \langle g_f(x)w(x - M), \nu \rangle$, is continuous, where $M = \int x\nu(\mathrm{d}x)$. So suppose $\lim_{n \to \infty} d_1(\nu_n, \nu) = 0$, and denote by M_n the mean of ν_n, i.e. $M_n = \int x\nu_n(\mathrm{d}x)$. We need to show that in this case

$$\lim_{n \to \infty} \left| \langle g_f(x)w(x - M_n), \nu_n \rangle - \langle g_f(x)w(x - M), \nu \rangle \right| = 0.$$

We use the triangle inequality to arrive at

$$\left| \langle g_f(x)w(x - M_n), \nu_n \rangle - \langle g_f(x)w(x - M), \nu \rangle \right| \leq \left| \langle g_f(x)\left[w(x - M_n) - w(x - M)\right], \nu_n \rangle \right|$$
$$+ \left| \langle g_f(x)w(x - M), \nu_n \rangle - \langle g_f(x)w(x - M), \nu \rangle \right|.$$

The first term on the right-hand side can be bounded above by $2a'|M_n - M|$. This term goes to zero as $n \to \infty$, due to $\lim_{n \to \infty} d_1(\nu_n, \nu) = 0$ and the equivalence of (1) and (2) in Lemma 6.1. For every $f \in H$, $g_f(x)w(x - M)$ is a bounded and continuous function, and consequently the second term goes to zero as $n \to \infty$ as well, due to $\lim_{n \to \infty} d_1(\nu_n, \nu) = 0$ and the equivalence of (1) and (2) in Lemma 6.1. Thus h_f is continuous for every $f \in H$.

Consequently the continuous mapping theorem and Lemma 6.7 together imply that

$$\langle g_f(x)w(x - m_n(\cdot)), \mu_n(\cdot) \rangle \Rightarrow_n \langle g_f(x)w(x - m(\cdot)), \mu(\cdot) \rangle$$

in $D([0, \infty), \mathbb{R})$. Lemma 6.8 then implies that

$$\int_0^\cdot \langle g_f(x)w(x - m_n(s)), \mu_n(s) \rangle \mathrm{d}s \Rightarrow_n \int_0^\cdot \langle g_f(x)w(x - m(s)), \mu(s) \rangle \mathrm{d}s$$

in $D([0, \infty), \mathbb{R})$. The integral function on the right-hand side is continuous, which implies that the one-dimensional distributions converge weakly in $\mathbb{R}$, i.e. (6.17) holds. $\qquad \square$

Proof of Corollary 6.13. Fix $t \geq 0$ and $f \in H$. Theorem 6.11 implies that

$$A_{t,f}(\mu_n(\cdot)) \Rightarrow_n 0$$

in $\mathbb{R}$. On the other hand, Theorem 6.12 implies that

$$A_{t,f}(\mu_n(\cdot)) \Rightarrow_n A_{t,f}(\mu(\cdot))$$

in $\mathbb{R}$. However, we know that a sequence cannot converge weakly to two different limits (Billingsley [8], Section 2), and hence $A_{t,f}(\mu(\cdot)) = 0$ almost surely. $\qquad \square$

6.5. *Uniqueness of the solution to the mean field equation*

In this subsection we prove Theorem 6.14 and Corollary 6.15.

Proof of Theorem 6.14. We have to show that there exists a constant c such that (6.11) holds. In order to do this we prove that there exists a constant c such that

$$d_H(t) \leq d_H(0) + c \int_0^t d_H(s) \, \mathrm{d}s, \tag{6.18}$$

from which (6.11) follows using Grönwall's lemma.

So let us now prove (6.18). Denote by $m^1(t)$ and $m^2(t)$ the mean of $\mu^1(t)$ and $\mu^2(t)$ respectively: $m^i(t) = \int x\mu^i(t, \mathrm{d}x)$ for $i = 1, 2$. We know that $\mu^1(\cdot)$ and $\mu^2(\cdot)$ are solutions to the mean field equation, with initial conditions μ_0^1 and μ_0^2 respectively, and so

$$A_{t,f}\big(\mu^i(\cdot)\big) = 0$$

holds for all $f \in H$, for $i = 1, 2$ (recall (3.8)). Using this, the definition of the metric d_H, and simple inequalities, we have

$$d_H(t) = \sup_{f \in H} \big|\langle f, \mu^1(t)\rangle - \langle f, \mu^2(t)\rangle\big|$$

$$\leq \sup_{f \in H} \big|\langle f, \mu_0^1\rangle - \langle f, \mu_0^2\rangle\big| + \int_0^t \sup_{f \in H} \big|\langle \big(\mathbb{E}\big(f(x + Z)\big) - f(x)\big)w\big(x - m^1(s)\big), \mu^1(s)\rangle$$

$$- \langle \big(\mathbb{E}\big(f(x + Z)\big) - f(x)\big)w\big(x - m^2(s)\big), \mu^2(s)\rangle\big| \, \mathrm{d}s.$$

The first term on the right-hand side is exactly $d_H(0)$, and so what we have to show is that the supremum inside the integral is at most a constant multiple of $d_H(s)$. As in the proof of Theorem 6.12, define

$$g_f(x) := \big(\mathbb{E}\big(f(x + Z)\big) - f(x)\big)$$

in order to abbreviate notation. We know that $g_{\mathrm{Id}} = \mathbb{E}(Z) = 1$ and that for every $f \in C_b$, $|f| \leq 1$, g_f is continuous and $|g_f| \leq 2$. First we use the triangle inequality:

$$\sup_{f \in H} \big|\langle g_f(x)w\big(x - m^1(s)\big), \mu^1(s)\rangle - \langle g_f(x)w\big(x - m^2(s)\big), \mu^2(s)\rangle\big|$$

$$\leq \sup_{f \in H} \big|\langle g_f(x)w\big(x - m^1(s)\big), \mu^1(s)\rangle - \langle g_f(x)w\big(x - m^1(s)\big), \mu^2(s)\rangle\big|$$

$$+ \sup_{f \in H} \big|\langle g_f(x)w\big(x - m^1(s)\big), \mu^2(s)\rangle - \langle g_f(x)w\big(x - m^2(s)\big), \mu^2(s)\rangle\big|.$$

We bound the two terms separately by a constant multiple of $d_H(s)$. Let us start with the first term. If $f \in C_b$, $|f| \leq 1$, then $g_f(x)w(x - m^1(s))$ is continuous in x and its absolute value is at most $2a$, so consequently the expression in the supremum is at most $2ad_H(s)$. If $f = \mathrm{Id}$, then the expression in the supremum is $|\langle w(x - m^1(s)), \mu^2(s)\rangle - \langle w(x - m^1(s)), \mu^1(s)\rangle|$ which is at most $ad_H(s)$ since $w(x - m^1(s))$ is continuous in x and $|w| \leq a$. Thus the first term is at most $2ad_H(s)$.

Now let us look at the second term. Using the fact that w' is bounded we have

$$\big|w\big(x - m^1(s)\big) - w\big(x - m^2(s)\big)\big| \leq a'\big|m^1(s) - m^2(s)\big| \leq a'd_H(s),$$

and since $|g_f| \leq 2$ for all $f \in H$, it follows that the second term is at most $2a'd_H(s)$. Therefore this shows (6.18) with $c := 2a + 2a'$. □

Proof of Corollary 6.15. If $d_H(0) = 0$ then, due to Theorem 6.14, $d_H(t) = 0$ for all $t \geq 0$, and consequently $\mu^1(t) = \mu^2(t)$ for all $t \geq 0$, so $\mu^1(\cdot) = \mu^2(\cdot)$. □

6.6. *Proof of Theorem 6.5*

Before proving Theorem 6.5, let us present a result for general Polish space state spaces that we use during the proof. Suppose E is a Polish space and d is a complete metric on E. For $x(\cdot) \in D([0, \infty), E)$, $T > 0$, and $\delta > 0$, let

$$w\big(x(\cdot), \delta, T\big) := \sup_{0 \leq s, t \leq T, |s-t| \leq \delta} d\big(x(t), x(s)\big).$$

(We note that this w should not be confused with our jump rate function w – the distinction will be obvious from the context.) Then:

Theorem 6.16. *Suppose* $\{X_n(\cdot)\}_{n \geq 1}$ *is a sequence of cadlag* E-*valued processes, where* E *is Polish.* $\{X_n(\cdot)\}_{n \geq 1}$ *is* C-*relatively compact in* $D([0, \infty), E)$ *if and only if the following two conditions hold*:

(a) *For every* $T > 0$ *and every* $\varepsilon > 0$ *there exists a compact set* $K^0 = K^0_{T,\varepsilon} \subset E$ *such that*

$$\sup_n \mathbb{P}\big(X_n(t) \notin K^0 \text{ for some } t \leq T\big) \leq \varepsilon.$$

(b) *For every* $T > 0$ *and every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\limsup_{n \to \infty} \mathbb{P}\big(w\big(X_n(\cdot), \delta, T\big) \geq \varepsilon\big) \leq \varepsilon.$$

This theorem follows from Corollary 3.7.4, Remark 3.7.3, and Theorem 3.10.2 of Ethier and Kurtz [13].

Note. (*a*) *is called the compact containment condition.*

Proof of Theorem 6.5. To prove that $\{P_n(\cdot)\}_{n \geq 1}$ is C-relatively compact in $D([0, \infty), \mathcal{P}_1(\mathbb{R}))$ we need to show that conditions (a) and (b) of Theorem 6.16 follow from conditions (i) and (ii) of Theorem 6.5. Notice that it is enough to demonstrate this for small ε's. First, let us start with the compact containment condition. Let $T > 0$ and $\varepsilon \in (0, \frac{1}{3})$ be fixed. To this $T > 0$, there exists a $c = c(T)$ and a $\tau = \tau(T) \in (0, 1)$ given by condition (i). For every $m \geq 1$, let $\varepsilon_m := \tilde{c}\varepsilon/m^{2/\tau}$, where $\tilde{c}^{-1} = \sum_{m \geq 1} m^{-2/\tau}$ so that $\sum_{m \geq 1} \varepsilon_m = \varepsilon$. Now by condition (i), to the fixed T and ε_m there exists a $K_m := K_{T,\varepsilon_m} \in \mathbb{R}$ such that

$$K_m \leq \frac{c}{\varepsilon_m^{1-\tau}} = \frac{c}{(\tilde{c}\varepsilon)^{1-\tau}} m^{2/\tau - 2}$$

and

$$\sup_n \mathbb{P}\Big(\sup_{0 \leq t \leq T} P_n\big(t, (-\infty, -K_m) \cup (K_m, \infty)\big) > \varepsilon_m\Big) < \varepsilon_m.$$

This implies that

$$\sup_n \mathbb{P}\Big(\sup_{0 \leq t \leq T} P_n\big(t, (-\infty, -K_m) \cup (K_m, \infty)\big) > 1/m^{2/\tau}\Big) < \varepsilon_m,$$

since $\varepsilon_m < 1/m^{2/\tau}$. Now given the K_m's, we define the following set of probability measures:

$$C^0 := \big\{\mu \in \mathcal{P}_1(\mathbb{R}) : \mu\big((-\infty, -K_m) \cup (K_m, \infty)\big) \leq 1/m^{2/\tau} \text{ for every } m \geq 1\big\}.$$

(Of course $C^0 = C^0_{T,\varepsilon}$.) The choice of the K_m's implies that

$$\sup_n \mathbb{P}\big(P_n(t) \notin C^0 \text{ for some } t \leq T\big) \leq \varepsilon,$$

since

$$\mathbb{P}\big(P_n(t) \notin C^0 \text{ for some } t \leq T\big) = \mathbb{P}\big(\exists m \geq 1, \exists t \geq T : P_n\big(t, (-\infty, -K_m) \cup (K_m, \infty)\big) > 1/m^{2/\tau}\big)$$

$$= \mathbb{P}\Big(\exists m \geq 1 : \sup_{0 \leq t \leq T} P_n\big(t, (-\infty, -K_m) \cup (K_m, \infty)\big) > 1/m^{2/\tau}\Big)$$

$$\leq \sum_{m=1}^{\infty} \mathbb{P}\Big(\sup_{0 \leq t \leq T} P_n\big(t, (-\infty, -K_m) \cup (K_m, \infty)\big) > 1/m^{2/\tau}\Big)$$

$$\leq \sum_{m=1}^{\infty} \varepsilon_m = \varepsilon.$$

Consequently,

$$\sup_n \mathbb{P}\big(P_n(t) \notin K^0 \text{ for some } t \le T\big) \le \varepsilon,$$

for $K^0 := \overline{C^0}$. This K^0 will be good as the compact set needed in condition (a) – what is left to show is that K^0 is compact in $\mathcal{P}_1(\mathbb{R})$. For this, it is enough to show that C^0 is relatively compact in $\mathcal{P}_1(\mathbb{R})$. We know (see, for instance, Feng and Kurtz [14], Appendix D) that $C^0 \subset \mathcal{P}_1(\mathbb{R})$ is relatively compact if and only if

$$\lim_{N \to \infty} \sup_{\mu \in C^0} \int_{|x| > N} |x| \mu(\mathrm{d}x) = 0. \tag{6.19}$$

So let us now check (6.19). Note that this is where the original form of Perkins' theorem [26], Theorem II.4.1 is not enough and we have to use our modified version, Theorem 6.5. If $N > K_m$, then

$$\sup_{\mu \in C^0} \int_{|x| > N} |x| \mu(\mathrm{d}x) \le \sup_{\mu \in C^0} \sum_{l=m}^{\infty} \int_{[-K_{l+1}, -K_l] \cup (K_l, K_{l+1}]} |x| \mu(\mathrm{d}x)$$

$$\le \sup_{\mu \in C^0} \sum_{l=m}^{\infty} K_{l+1} \mu\big((-\infty, -K_l) \cup (K_l, \infty)\big)$$

$$\le \sum_{l=m}^{\infty} \frac{c}{(\tilde{c}\varepsilon)^{1-\tau}} (l+1)^{2/\tau - 2} \frac{1}{l^{2/\tau}}.$$

This last expression goes to 0 as $m \to \infty$, which shows (6.19), and consequently we have shown that condition (i) implies condition (a).

Now let us show that conditions (i) and (ii) imply condition (b). First we show that for every $f \in C_b(\mathcal{P}_1(\mathbb{R}))$, $\{f \circ P_n(\cdot)\}_{n \ge 1}$ is C-relatively compact in $D([0, \infty), \mathbb{R})$. By the characterization of C-relatively compactness we need to check conditions (a) and (b) of Theorem 6.16. Due to the boundedness of f, (a) is immediate. Now let us show (b). Fix $f \in C_b(\mathcal{P}_1(\mathbb{R}))$, $T > 0$, and $\varepsilon > 0$. Choose $K^0 = K^0_{T, \varepsilon}$ as in (a). Define

$$A = \left\{ h : \mathcal{P}_1(\mathbb{R}) \to \mathbb{R} \,\middle|\, h(\mu) = \sum_{i=1}^{k} a_i \prod_{j=1}^{m_i} \langle f_{i,j}, \mu \rangle, a_i \in \mathbb{R}, f_{i,j} \in C_b, k, m_i \in \mathbb{Z}_+ \right\} \subset C_b\big(\mathcal{P}_1(\mathbb{R})\big).$$

Then A is an algebra containing the constant functions and separating points in $\mathcal{P}_1(\mathbb{R})$. So by the Stone–Weierstrass theorem (see for instance Willard [33]) there exists an $h \in A$ such that

$$\sup_{\mu \in K^0} \big|h(\mu) - f(\mu)\big| < \varepsilon.$$

If $\{Y_n(\cdot)\}_{n \ge 1}$ and $\{Z_n(\cdot)\}_{n \ge 1}$ are C-relatively compact in $D([0, \infty), \mathbb{R})$ then so are $\{aY_n(\cdot) + bZ_n(\cdot)\}_{n \ge 1}$ and $\{Y_n(\cdot)Z_n(\cdot)\}_{n \ge 1}$ for any $a, b \in \mathbb{R}$ (see the characterization of Theorem 6.16). Therefore condition (ii) of the theorem implies that $\{h \circ P_n(\cdot)\}_{n \ge 1}$ is C-relatively compact in $D([0, \infty), \mathbb{R})$. Now using the characterization of Theorem 6.16 again, there exists a $\delta > 0$ such that

$$\limsup_{n \to \infty} \mathbb{P}\big(w\big(h \circ P_n(\cdot), \delta, T\big) \ge \varepsilon\big) \le \varepsilon.$$

To abbreviate notation, we introduce $P_n([0, T]) := \{P_n(t) : t \in [0, T)\}$. If $s, t \le T$, then

$$\big|f\big(P_n(t)\big) - f\big(P_n(s)\big)\big| = \big|f\big(P_n(t)\big) - f\big(P_n(s)\big)\big| \mathbf{1}\big[P_n([0, T]) \subseteq K^0\big]$$

$$+ \big|f\big(P_n(t)\big) - f\big(P_n(s)\big)\big| \mathbf{1}\big[P_n([0, T]) \nsubseteq K^0\big]$$

$$\leq 2 \sup_{\mu \in K^0} \big|h(\mu) - f(\mu)\big| + \big|h\big(P_n(t)\big) - h\big(P_n(s)\big)\big|$$

$$+ 2\|f\|_\infty \mathbf{1}\big[P_n\big([0, T]\big) \nsubseteq K^0\big]$$

and so by the definition of $w(f \circ P_n(\cdot), \delta, T)$ we have

$$w\big(f \circ P_n(\cdot), \delta, T\big) \leq 2\varepsilon + w\big(h \circ P_n(\cdot), \delta, T\big) + 2\|f\|_\infty \mathbf{1}\big[P_n\big([0, T]\big) \nsubseteq K^0\big].$$

Therefore

$$\limsup_{n \to \infty} \mathbb{P}\big(w\big(f \circ P_n(\cdot), \delta, T\big) \geq 3\varepsilon\big) \leq \limsup_{n \to \infty} \mathbb{P}\big(P_n\big([0, T]\big) \nsubseteq K^0\big)$$

$$+ \limsup_{n \to \infty} \mathbb{P}\big(w\big(h \circ P_n(\cdot), \delta, T\big) \geq \varepsilon\big)$$

$$\leq \varepsilon + \varepsilon = 2\varepsilon < 3\varepsilon,$$

by the choice of K^0 and the fact that $\{h \circ P_n(\cdot)\}_{n \geq 1}$ is C-relatively compact in $D([0, \infty), \mathbb{R})$.

So let us recap what we have done up until now: we have shown that (i) implies the compact containment condition and that (i) and (ii) imply that for every $f \in C_b(\mathcal{P}_1(\mathbb{R}))$, $\{f \circ P_n(\cdot)\}_{n \geq 1}$ is C-relatively compact in $D([0, \infty), \mathbb{R})$. These imply that $\{P_n(\cdot)\}_{n \geq 1}$ is relatively compact in $D([0, \infty), \mathbb{R})$ (see Theorem 3.9.1 of Ethier and Kurtz [13]). What remains is to show that it is not only relatively compact, but also C-relatively compact.

We now show that (i) and (ii) imply (b). Let us define the metric $d := \min\{d_1, 1\}$ and let

$$w_d\big(P_n(\cdot), \delta, T\big) := \sup_{0 \leq s, t \leq T, |s-t| \leq \delta} d\big(P_n(t), P_n(s)\big).$$

Let $T > 0$ and $\varepsilon > 0$ be fixed, and choose K^0 as in (a) again. Since K^0 is compact, we can choose $\mu_i \in K^0$, $i = 1, \ldots, M$, so that $K^0 \subseteq \bigcup_{i=1}^M B(\mu_i, \varepsilon)$, where

$$B(\mu, \varepsilon) = \big\{\nu \in \mathcal{P}_1(\mathbb{R}): d_1(\mu, \nu) < \varepsilon\big\}.$$

Since $\varepsilon < 1$,

$$B(\mu, \varepsilon) = \big\{\nu \in \mathcal{P}_1(\mathbb{R}): d(\mu, \nu) < \varepsilon\big\}.$$

Now let $f_i(\mu) := d(\mu_i, \mu)$. Clearly $f_i \in C_b(\mathcal{P}_1(\mathbb{R}))$. We have proved that for every $f \in C_b(\mathcal{P}_1(\mathbb{R}))$, $\{f \circ P_n(\cdot)\}_{n \geq 1}$ is C-relatively compact in $D([0, \infty), \mathbb{R})$, which implies that there exists a $\delta > 0$ such that

$$\sum_{i=1}^M \limsup_{n \to \infty} \mathbb{P}\big(w\big(f_i \circ P_n(\cdot), \delta, T\big) \geq \varepsilon\big) \leq \varepsilon. \tag{6.20}$$

If $\mu, \nu \in K^0$, choose μ_j so that $d(\nu, \mu_j) \leq \varepsilon$. Then

$$d(\mu, \nu) \leq d(\mu, \mu_j) + d(\mu_j, \nu) \leq \big|d(\mu, \mu_j) - d(\mu_j, \nu)\big| + 2d(\mu_j, \nu) \leq \max_i \big|f_i(\mu) - f_i(\nu)\big| + 2\varepsilon.$$

Now let $s, t \leq T$; the above inequality implies

$$d\big(P_n(t), P_n(s)\big) = d\big(P_n(t), P_n(s)\big)\mathbf{1}\big[P_n\big([0, T]\big) \subseteq K^0\big] + d\big(P_n(t), P_n(s)\big)\mathbf{1}\big[P_n\big([0, T]\big) \nsubseteq K^0\big]$$

$$\leq \max_i \big|f_i \circ P_n(t) - f_i \circ P_n(s)\big| + 2\varepsilon + \mathbf{1}\big[P_n\big([0, T]\big) \nsubseteq K^0\big],$$

where we used that $d \leq 1$. Consequently

$$w_d\big(P_n(\cdot), \delta, T\big) \leq \max_i w\big(f_i \circ P_n(\cdot), \delta, T\big) + 2\varepsilon + \mathbf{1}\big[P_n\big([0, T]\big) \nsubseteq K^0\big].$$

Now note that $\{w(P_n(\cdot), \delta, T) \geq 3\varepsilon\}$ and $\{w_d(P_n(\cdot), \delta, T) \geq 3\varepsilon\}$ are exactly the same events, and consequently

$$
\begin{aligned}
\limsup_{n\to\infty} \mathbb{P}\big(w\big(P_n(\cdot), \delta, T\big) \geq 3\varepsilon\big) &= \limsup_{n\to\infty} \mathbb{P}\big(w_d\big(P_n(\cdot), \delta, T\big) \geq 3\varepsilon\big) \\
&\leq \limsup_{n\to\infty} \mathbb{P}\Big(\max_i w\big(f_i \circ P_n(\cdot), \delta, T\big) \geq \varepsilon\Big) + \limsup_{n\to\infty} \mathbb{P}\big(P_n\big([0,T]\big) \nsubseteq K^0\big) \\
&\leq \limsup_{n\to\infty} \sum_{i=1}^{M} \mathbb{P}\big(w\big(f_i \circ P_n(\cdot), \delta, T\big) \geq \varepsilon\big) + \limsup_{n\to\infty} \mathbb{P}\big(P_n\big([0,T]\big) \nsubseteq K^0\big) \\
&\leq \varepsilon + \varepsilon = 2\varepsilon < 3\varepsilon,
\end{aligned}
$$

using (6.20) and the choice of K^0. This shows condition (b), and therefore completes the proof. $\qquad\square$

Acknowledgments

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